Asymptotics of N-dimensional tori in the generalized Korteweg – de Vries equation

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Abstract. We consider the generalized Korteweg – de Vries (KdV) equation with periodic boundary condition by space variable. The asymptotics of periodic solutions and invariant tori are considered in a sufficiently small neighbourhood of the zero equilibrium state. There are considered two cases: the coefficient of the quadratic term in the right part of generalized KdV equation is non-zero or zero. In the first case, the cycles and two-dimension tori are unstable. In the second, case it is proved that cycles are stable and tori are unstable.

1. Introduction

Let us consider the boundary problem (see [1, 2])

$$u_t + u_{xxx} + (\alpha u + \beta u^2)u_x = au + bu^2 + cu^3, \tag{1}$$

$$u(t, x + T) = u(t, x), \tag{2}$$

where $u(t,x) \in W_2^3$ for all $t > 0$, parameters α, β, a, b, c, T are constants.

Note that the position of the characteristic (wave) equation roots

$$\lambda_k = ik^3 + a \tag{3}$$

is important for problem of the zero solution stability. By (2), the value of k in (3) is $2\pi n/T$, where n is integer. If $a < 0$ then we have $\operatorname{Re}\lambda_k = a < 0$ and all solutions of (1), (2) from a sufficiently small neighbourhood of the zero equilibrium state tend to zero as $t \to \infty$. If $a > 0$ then $\operatorname{Re}\lambda_k > 0$ and the problem stops to be local. Suppose that the parameter a is small enough:

$$a = \varepsilon a_0,\ 0 < \varepsilon \ll 1. \tag{4}$$

Let us investigate the question about a construction periodic solutions and invariant tori of different dimension in a small enough neighbourhood of zero equilibrium state. The condition (4) allows us to use the methods of bifurcation analysis [3, 4, 5]. Infinitely many roots of the characteristic equation (3) tend to imaginary axis as $\varepsilon \to 0$. Thus, the critical case of infinite dimension occurs in the problem of zero solution stability for (1), (2). We recall that well-known methods of local invariant integral manifolds (see, for example, [6, 7, 8]) and methods of normal forms (see [9, 10]) are used for investigation of the local dynamics of the equation in the finite-dimensional critical case. The methods permit us to study a special finite-dimensional non-linear system of ordinary differential equations (ODE) instead of the equation. It is impossible to pass boundary problem (1), (2), (4) to a finite-dimensional system of ODE. However it is possible to obtain the results about the existence of equilibrium states, periodic solutions, and invariant tori of various dimensions in a small neighbourhood of zero using the formal method of normal forms. It is possible to explore the question of their stability. We show this bellow.

Section 2 contains the formal problem statement. In sections 3 and 4, we construct the asymptotics of invariant tori. Moreover, in section 3, we study the general case $b \neq 0$ and give a result about the non-stability of cycles and two-dimension tori. Section 4 presents more simple situation $b = 0$. In this case, it is proved that there exists stable cycles and unstable tori.

2. Problem statement

Let N be a fix natural number. We are interested in the construction of the N-dimension torus asymptotics in a sufficiently small neighbourhood of zero. We fix different natural $n_1, \ldots, n_N$. Let

$$k_1 = 2\pi n_1/T, \ldots, k_N = 2\pi n_N/T. \tag{5}$$

We find the solution of (1), (2) in the form of the series

$$u = \eta(t,\varepsilon) + \sum_{j=1}^{N} (\xi_j(t,\varepsilon)\exp(ik_jx + ik_j^3 t) + c\bar{c}) + \ldots, \tag{6}$$

where the symbol $c\bar{c}$ means the term which is complex conjugated to the previous one. The terms are trigonometric polynomials from t and x. Here functions $\eta = \eta(t,\varepsilon)$, $\xi_j = \xi_j(t,\varepsilon)$ and their derivatives are small enough. The last ones are represented by η, ξ_j and ε:

$$\begin{aligned} \dot\eta &= \alpha_1\varepsilon\eta + \alpha_2\eta^2 + \sum_{j=1}^{N}\alpha_{3j}|\xi_j|^2 + \alpha_4\eta^3 + \eta\sum_{j=1}^{N}\alpha_{5j}|\xi_j|^2, \\ \dot\xi_j &= \beta_{1j}\varepsilon\xi_j + \beta_{2j}\eta\xi_j + \xi_j\Big(\sum_{s=1}^{N}\beta_{3js}|\xi_s|^2 + \beta_{4j}\eta^2\Big), \ j = 1,\ldots,N. \end{aligned} \tag{7}$$

Let the system (7) be called a normal form. Here we use the terminology of the classical theory of bifurcation analysis.

The following sequence of actions is standard for similar problems [11, 12, 13]. Put (6) to (1) instead of u. Change $\dot\eta$ and $\dot\xi_j$ to (7) (coefficients α_m, α_{3j}, α_{5j}, β_{mj}, β_{3js}, $m = 1,2,4$, $j,s = 1,\ldots,N$ are unknown). Then we equate the coefficients of the same powers of ε, η and ξ_j. As a result we obtain values of α_m, α_{3j}, α_{5j}, β_{mj} from a resolvability condition. Then we substitute $\rho_j\exp(i\varphi_j)$, $\rho_j \geq 0$, for ξ_j, $j = 1,\ldots,N$, in (7) to find automodel periodic solution. We obtain the amplitude system, which separates from (7). It is important whether the parameter b is zero or not for learning the amplitude system.

3. Case $b \neq 0$

If $b \neq 0$, we obtain the following values of coefficients of normal form (7) for $j,s = 1,\ldots,N$:

$$\begin{gathered} \alpha_1 = a_0,\ \alpha_2 = b,\ \alpha_{3j} = 2b,\ \alpha_4 = c,\ \alpha_{5j} = 6c, \\ \beta_{1j} = a_0,\ \beta_{2j} = 2b - i\alpha k_j,\ \beta_{3jj} = 3c + \tfrac{\alpha b}{2k_j^2} + i\Big(\tfrac{b^2}{3k_j^3} - \tfrac{\alpha^2}{6k_j} - \beta k_j\Big), \\ \beta_{3js} = 6c - \tfrac{4b\alpha}{3(k_j^2-k_s^2)} + i\Big(\tfrac{8b^2}{3k_j(k_j^2-k_s^2)} - 2\beta k_j\Big) \text{ where } s \neq j,\ \beta_{4j} = 3c - i\beta k_j. \end{gathered} \tag{8}$$

The substitution $\xi = \rho_j \exp(i\varphi_j)$ permits one to pass to the amplitude system

$$\begin{aligned} \dot{\eta} &= a_0\varepsilon\eta + 2b \sum_{s=1}^{N} \rho_s^2 + b\eta^2 + c\eta^3 + 6c\eta \sum_{s=1}^{N} \rho_s^2, \\ \dot{\rho}_j &= \rho_j(a_0\varepsilon + 2b\eta + \sum_{s=1}^{N} \operatorname{Re}\beta_{3js}\rho_s^2 + 3c\eta^2), \ j = 1, \ldots, N. \end{aligned} \tag{9}$$

If $c \neq 0$ then system (9) has three equilibrium points such that $\rho_1 = \ldots = \rho_N = 0$:

1) $\eta = \rho_1 = \ldots = \rho_N = 0$,
2) $\eta = -2a_0c\varepsilon/b + O(\varepsilon^2)$, $\rho_1 = \ldots = \rho_N = 0$,
3) $\eta = -b/c + O(\varepsilon)$, $\rho_1 = \ldots = \rho_N = 0$.

The equilibrium points in the first and second cases have different stability. If $a_0 > 0$, case 1) is unstable and 2) is unstable. (And conversely.) In the last case, the equilibrium state is not in a sufficiently small neighbourhood of zero. Learning this case is not a subject of our paper.

It is not difficult to prove the following statement for local case 2).

Lemma 1. *For all sufficiently small $\varepsilon > 0$, $a_0 > 0$, $b \neq 0$, $c \neq 0$, there exists an equilibrium stable state*

$$u(t, x) = -\frac{2a_0c\varepsilon}{b}\varepsilon + O(\varepsilon^2), \tag{10}$$

which is asymptotic by the residual with solutions of (1), (2).

Then let us consider the case $N = 2$. Suppose that η and ρ_j have ε order. Thus, the equilibrium points of (9) are given at table 1.

Table 1. Equilibrium points of (9) in case $N = 2$.

η	$\rho_1 \geq 0$	$\rho_2 \geq 0$	Stability
0	0	0	U
$-\frac{a_0\varepsilon}{b} + O(\varepsilon^2)$	0	0	S
$-\frac{a_0\varepsilon}{2b} + O(\varepsilon^2)$	$\frac{a_0\varepsilon}{2\sqrt{2}\lvert b\rvert} + O(\varepsilon^2)$	0	U
$-\frac{a_0\varepsilon}{2b} + O(\varepsilon^2)$	0	$\frac{a_0\varepsilon}{2\sqrt{2}\lvert b\rvert} + O(\varepsilon^2)$	U
$-\frac{a_0\varepsilon}{2b} + O(\varepsilon^2)$	$\rho_{12*}\varepsilon + O(\varepsilon^2)$	$\rho_{21*}\varepsilon + O(\varepsilon^2)$	U

Here (and bellow in the next section) U means "unstable", S means "stable" and

$$\rho_{js*} = \frac{a_0}{2\sqrt{2}|b|}\sqrt{\frac{3c - 4\alpha b/(3(k_j^2 - k_s^2)) - \alpha b/(2k_s^2)}{6c - \alpha b/(2k_j^2) - \alpha b/(2k_s^2)}}. \tag{11}$$

Lemma 2. *For all sufficiently small $\varepsilon > 0$, $a_0 > 0$, $k_1 = 2\pi n_1/T$, $k_2 = 2\pi n_2/T$, where $n_1 \neq n_2$ are natural, and b, c, α such that numbers*

$$3c - 4\alpha b/(3(k_1^2 - k_2^2)) - \alpha b/(2k_2^2), \quad 3c - 4\alpha b/(3(k_2^2 - k_1^2)) - \alpha b/(2k_1^2) \tag{12}$$

are non-zero and have the same sign, there exists two-dimension unstable torus

$$u(t, x) = -\frac{a_0}{2b}\varepsilon + \sum_{j=1}^{2} \left(\rho_{j*} \exp\left(ik_jx + i(k_j^3 + \varphi_{j*})t\right) + c\overline{c}\right) + O(\varepsilon^2), \tag{13}$$

which are asymptotic by the residual with solutions of (1), (2). Here

$$\rho_{1*} = \rho_{12*}\varepsilon + O(\varepsilon^2), \quad \rho_{2*} = \rho_{21*}\varepsilon + O(\varepsilon^2), \quad \varphi_{j*} = \frac{\alpha k_j a_0 \varepsilon}{2b} + O(\varepsilon^2), \ j = 1, 2, \tag{14}$$

ρ_{js*} *is defined by (11).*

The similar result is proved in a more simple case when only one of ρ_j is non-zero. Such cycles are also unstable.

4. Case $b = 0$

In the case, we obtain the following coefficients of normal form (7) for $j, s = 1, \ldots, N$:

$$\alpha_1 = \beta_{1j} = a_0, \ \alpha_2 = \alpha_{3j} = 0, \ \alpha_4 = c, \ \alpha_{5j} = 6c, \ \beta_{2j} = -i\alpha k_j,$$

$$\beta_{3js} = 6c - 2i\beta k_j \text{ where } s \neq j, \ \beta_{3jj} = 3c - i(\alpha^2/(6k_j) + \beta k_j), \ \beta_{4j} = 3c - i\beta k_j.$$

Thus, the substitution $\xi_j = \rho_j \exp(i\varphi_j t)$ leads the system (7) to the following amplitude system

$$\begin{aligned} \dot{\eta} &= \eta(a_0\varepsilon + 6c \sum_{s=1}^{N} \rho_s^2 + c\eta^2), \\ \dot{\rho}_j &= \rho_j(a_0\varepsilon + 6c \sum_{s=1}^{N} \rho_s^2 - 3c\rho_j^2 + 3c\eta^2), \ j = 1, \ldots, N. \end{aligned} \tag{15}$$

Let us find equilibrium states of (15). Suppose that $a_0 > 0$, $c < 0$.

Firstly, note that there exists zero equilibrium point and an equilibrium point such that $\rho_1 = \ldots = \rho_N = 0$ and η is non-zero. It is easy to prove constructing the Jacobian matrix that the first one is unstable. If η is non-zero, we have $\eta = \pm\sqrt{-a_0\varepsilon/(3c)}$. Both of the equilibrium states are stable.

Secondly, suppose that $\rho_{j_1}, \ldots, \rho_{j_L} > 0$, $L = 1, \ldots, N$, and $\rho_j = 0$ whenever $j \notin \{j_1, \ldots, j_L\}$. Let us consider separately two cases: $\eta = 0$ and $\eta \neq 0$.

If $\eta = 0$, we find $\rho_{j_s} = \sqrt{-a_0\varepsilon/(3(2L-1)c)}$, $s = 1, \ldots, L$. Such equilibrium states are stable just if $L = 1$. If $L > 1$, it is possible to prove that the Jacobian matrix of the system (15) has a positive eigenvalue.

If $\eta \neq 0$, we obtain equilibrium points of (15) such that $\eta = \pm\sqrt{-a_0\varepsilon/((4L+1)c)}$, $\rho_{j_s} = \sqrt{-2a_0\varepsilon/(3(4L+1)c)}$, $s = 1, \ldots, L$. All of them are unstable.

For example, we give the table 2.

As a result, we obtain the following

Theorem. *For all sufficiently small $\varepsilon > 0$ and for $a_0 > 0$, $c < 0$, $k_1 = 2\pi n_1/T, \ldots, k_N = 2\pi n_N/T$ (where $n_1, \ldots, n_N$ are different natural numbers), there exists stable equilibrium states*

$$u(t,x) = \pm\sqrt{-\frac{a_0\varepsilon}{c}} + O(\varepsilon),$$

a stable cycles

$$u(t,x) = \sqrt{-\frac{a_0\varepsilon}{3c}} \exp\left(ik_1 x + i(k_1^3 + \varphi_{j*})t\right) + c\bar{c} + O(\varepsilon),$$

and unstable tori

$$u(t,x) = \rho_{j*} \sum_{j=1}^{L} \left(\exp\left(ik_j x + i(k_j^3 + \varphi_{j*})t\right) + c\bar{c} \right) + O(\varepsilon), \ L = 2, \ldots, N,$$

Table 2. Equilibrium points of (15) in case $N = 2$.

η	$\rho_1 \geq 0$	$\rho_2 \geq 0$	Stability
0	0	0	U
$\pm\sqrt{-\frac{a_0\varepsilon}{c}}$	0	0	S
0	$\sqrt{-\frac{a_0\varepsilon}{3c}}$	0	S
0	0	$\sqrt{-\frac{a_0\varepsilon}{3c}}$	S
0	$\sqrt{-\frac{a_0\varepsilon}{9c}}$	$\sqrt{-\frac{a_0\varepsilon}{9c}}$	U
$\pm\sqrt{-\frac{a_0\varepsilon}{5c}}$	$\sqrt{-\frac{2a_0\varepsilon}{15c}}$	0	U
$\pm\sqrt{-\frac{a_0\varepsilon}{5c}}$	0	$\sqrt{-\frac{2a_0\varepsilon}{15c}}$	U
$\pm\sqrt{-\frac{a_0\varepsilon}{9c}}$	$\sqrt{-\frac{2a_0\varepsilon}{27c}}$	$\sqrt{-\frac{2a_0\varepsilon}{27c}}$	U

$$u(t,x) = \eta_{j*\pm} + \rho_{j*}\sum_{j=1}^{L}\Big(\exp\left(ik_jx + i(k_j^3 + \varphi_{j*\mp})t\right) + c\bar{c}\Big) + O(\varepsilon),\ \ L = 1,\ldots,N,$$

which are asymptotic by the residual with solutions of boundary problem (1), (2). Here

$$\rho_{j*} = \sqrt{\frac{-2a_0\varepsilon}{3(4L+1)c}}, \quad \eta_{j*\pm} = \pm\sqrt{\frac{-a_0\varepsilon}{(4L+1)c}},$$

$$\varphi_{j*} = \Big(\frac{\alpha^2}{6k_j} + 3\beta k_j\Big)\frac{a_0\varepsilon}{9c}, \quad \varphi_{j*\mp} = \mp\alpha k_j\sqrt{-\frac{a_0\varepsilon}{9c}} + O(\varepsilon).$$

5. Conclusion

The local dynamics of generalized KdV equation is studied using the methods based on the classical bifurcation theory. The question about the existence, asymptotics and stability of periodic solutions and tori of the boundary problem with periodic boundary condition is considered in two cases: when the parameter b of the quadratic non-linearity is zero and non-zero. In the first case, it is shown that, in condition of the positivity of the linear term in the right part ($a_0 > 0$), the cycles and two-dimension tori are unstable. In the second case, it is proved that, for $a_0 > 0$ and $c < 0$ (the condition guarantees the dissipativity of the problem), the cycles are stable and the tori are unstable.

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