Interfaces in evolutionary games

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Abstract. We investigate geometrical aspects of a spatial evolutionary game. The game is based on the Prisoner's dilemma. We analyze the geometrical structure of the space distribution of cooperators and defectors in the steady-state regime of evolution. We develop algorithm for the identification of the interfaces between clusters of cooperators and defectors, and measure fractal properties of the interfaces.

1. Introduction

In recent years much interest is paid to the investigation of physical and mathematical properties of interfaces in the plane and to identification of the origins of phase transitions. The interest is due to two main factors, the scientific factor and the practical factor. The first one is the development of the SLE (Stochastic Loewner Evolution [1], named later as Shcramm-Loewner Evolution) which connects properties of the random walk in the upper half-plane with the properties of Conformal Field Theory (CFT) [2]. It turns out to be the third alternative approach, complementing the algebraic solution [3] and the CFT solution [4], for solving critical properties of some two-dimensional systems of classical statistical physics. Among successful examples of this approach are the percolation problem [5, 6] and Ising model [7]. The second one is an interest to the investigation of physics and mathematics of the processes the planar geometry, connected to practicalities of the development of new materials for next generation of microelectronic devices [8].

In this paper we focus on the analysis of geometrical properties of interfaces emerging in the steady-state regime of the space-evolutionary game [9, 10]. The game is played by L^2 agents arranged on an $L{\times}L$ rectangular lattice with periodic boundary conditions. At each instant of time, an agent can be in a state of a cooperator $\mathcal{C}$, or a defector $\mathcal{D}$. Each agent computes their strategy to either keep the current state, or to change it depending on the payoff computed as the sum of the pair interaction with the neighbors. This is done for all players in parallel. The new state is assigned for all neighbors and time is incremented by one. It was found in [11] that starting with some initial distribution, system goes to the steady-state distribution of cooperators and defectors. They found also that starting from some regular distribution of players, regular self-similar structures appear, while random initial configurations lead to random steady-state structures, and the final steady-state stage depends on the payoff parameter.

It is natural to ask, what are the geometrical properties of the steady-state random structures? In particular, what is the fractal dimension of the interfaces between clusters of cooperators and

defectors? The answer can be informative for the possible classification of the changes of the steady-state regimes while tuning payoff parameter.

In the theory of phase transitions [12], first- and second-order transitions are most investigated. For a second-order transition in two spatial dimensions, the interface length $\mathcal{L}$ scales with the systems size L as a power law $\mathcal{L} \propto L^{\theta}$ [13]. For a first-order transition, the scaling of the interface of droplets is linear $\mathcal{L} \propto L$ [14]. In recent years, there is a discussion of the so-called mixed transitions, where some observables behave in a way expected for a second-order transition while other observables behave in a first-order way. So far, an analytic investigation of a mixed transition was done in a one-dimensional system [15]. In the usual percolation problem, the probability to belong to the infinite cluster behaves as power law of the parameter of the model. It was argued by some authors that in dynamical percolation, this probability can change sharply as a function of the parameter [16], which is is now known as discontinuous percolation [17]. We are not aware of any analysis of the interfaces in these novel phenomena.

In our model, the steady-state structures do percolate for some range of values of the payoff parameter, thus demonstrating a correlated behavior of players, which is similar to the case of a second-order phase transition. There are sharp changes of the observables while changing value of the payoff parameter, which is typical for first-order phase transition. At the same time, our model is not an equilibrium model of statistical mechanics, where one defines a thermodynamic equilibrium and fluctuations. The steady-state regime is achieved through a dynamic process, not some kind of an equilibration process. This way the steady-state structures are the self-organized.

The paper is organized as follows. In Section 2 we describe the model and review some previously known results. In Section 3 we describe algorithms we employ for simulations and analysis. Section 4 contains results of simulations and data analysis. In Section 5 we discuss results and future work.

2. Model of space evolution game

A prototypical model of the game theory is the so-called *Prisoner's dilemma* (PD), played by two players in discrete time steps. In each round of the game, each player uses one of two possible strategies, *cooperate*, $\mathcal{C}$, or *defect*, $\mathcal{D}$, and receives a payoff which depends on the strategies of the player and its opponent [18]. We use the following payoff structure [9, 11]: (i) If two players are $\mathcal{D}$, they receive nothing; (ii) If both players $\mathcal{C}$, each of them receives a payoff of S, which we set to $S = 1$ without loss of generality; (iii) In the interaction of $\mathcal{C}$ and $\mathcal{D}$, the $\mathcal{D}$ receives a payoff $T > S$ and the $\mathcal{C}$ receives zero. This way, the payoff structure only depends on a single parameter, the payoff parameter $b = T/S$.

The game starts with some initial configuration of $\mathcal{C}$ and $\mathcal{D}$ placed in the vertices of the square lattice with periodic boundary conditions, i.e. it is square lattice on a torus with $L \times L$ vertices. Each agent computes the payoff of its "interaction" with 8 neighbors (in a way chess king's moves) and with itself using the above rule for pair interaction. Payoff is calculated as the sum of the pair-wise payoffs. In the next time step (next generation, on the language of the game [11]), an agent adopts the strategy of those of its neighbors (including itself) that received the highest payoff. Thus the future state of an agent depends on the state of the cell itself, its 8 neighbors and their neighbors, i.e., 25 agents in total.

The dynamics of the game depends on the value of the payoff parameter b. Possible payoffs of agents can take only discrete values of the form $i + bj$, with i and j being non-negative integers. Therefore, the switches between different dynamic regimes can only occur at some discrete set of b values [11]. We concentrate our analysis for the narrow range of $3/2 < b < 2$, which we find the most interesting. This range includes several dynamic regimes, as seen in the left panel of Figure 1: the steady state average density of $\mathcal{C}$ changes abruptly at $b = 3/2, 8/5, 5/3, 7/4$
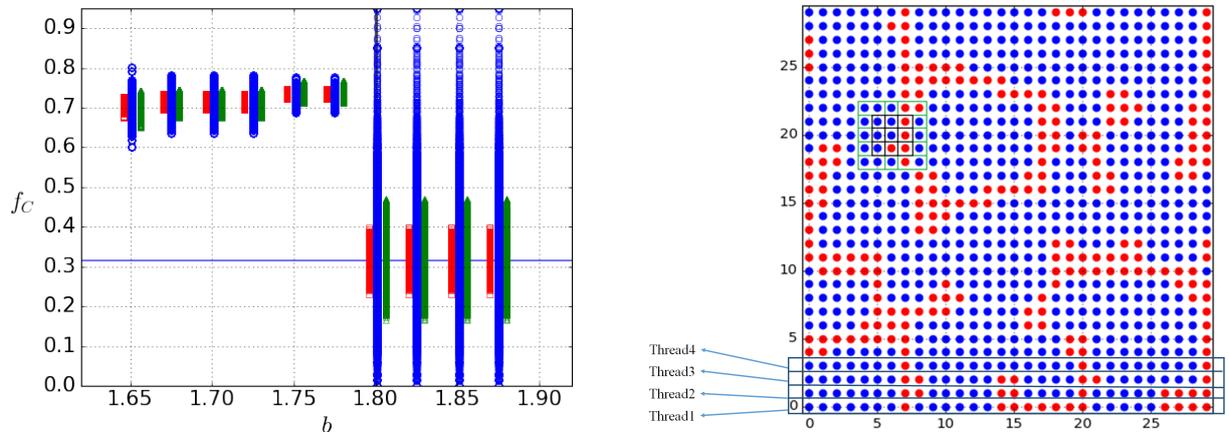
Figure 1. Left: Concentration of cooperators $\mathcal{C}$, f_C, as a function of the payoff parameter b for the lattice sizes 20x20 (blue), 50x50 (green), and 100x100 (red). For clarity, green signs are shifted horizontally slightly to the right and red signs are shifted to the left. All simulations are performed at $b = 1.651, 1.675, 1.701, 1.725, 1.751, 1.775, 1.801, 1.825, 1.851, 1.875$. Each point is a single measurement in a simulation of 2×10^4 generations of 25 independent realizations of initial conditions. Blue line denotes the magic value $f_C = 12 \log 2 - 8 \approx 0.318$ [11].
Right: Illustration of the association of the lattice vertices and threads. We also show the vertex with coordinates (6,20) surrounded by the neighbors (inside the magenta box) which influence its decision on the next time step.

and 9/5. These values can be explained microscopically by analyzing probabilities of stable oscillations of small $\mathcal{C}$-clusters for $b < 9/5$ and probabilities of growing $\mathcal{D}$-clusters for $b > 9/5$ following [11].

Data for the Figure 1-left was obtained from simulations starting with sufficiently dense and random initial configuration of $\mathcal{C}$ and $\mathcal{D}$, f.e. for concentrations $f_C = 0.9$ of cooperators and $f_D = 0.1$ for defectors. The spread of the computed steady-state concentration f_C in the figure can be clearly attributed to the finite size of the system, as red signs are spread less than green signs, and green signs is less than the blue signs, in accordance with the relation of the corresponding lattice sizes $100 > 50 > 20$.

Figure 2 shows representative snapshots of steady-state structures in the game field. It can be seen that in the left panel there is a blue path (corresponding to $\mathcal{C}$) from one side of the lattice to another one. In other words, a blue cluster percolates, and in the "thermodynamic limit" ($L \to \infty$) will be infinite. At the right panel, the red cluster percolates. Clearly, something happens at the critical value $b = 9/5$. It is known [11] that for $b > 9/5$ a square with nine $\mathcal{D}$ can grow in the sea of the $\mathcal{C}$. Due to the random initial condition, system self organizes into a steady state similar to the one show in the right panel of Figure 2, and the average concentration is just the magic value $f_C = 12 \log 2 - 8 \approx 0.318$ calculated for the 9 $\mathcal{D}$ cluster in Ref. [11].

3. Algorithms
We use OpenMP to accelerate simulations. Each row of the lattice is processed by an OpenMP thread as illustrated in the right panel of Figure 1. This provides a significant speed-up and allows us to simulate larger system sizes, see Fig. 3

We use the Hoshen-Kopelman algorithm [19], which provides the exact decomposition of the game field into connected clusters of $\mathcal{C}$ neighbors and $\mathcal{D}$ neighbors. Left panel of Fig. 4 illustrates the algorithm. We start with the upper-left site (0,0) and assign it to the $\mathcal{C}$ cluster number 1.
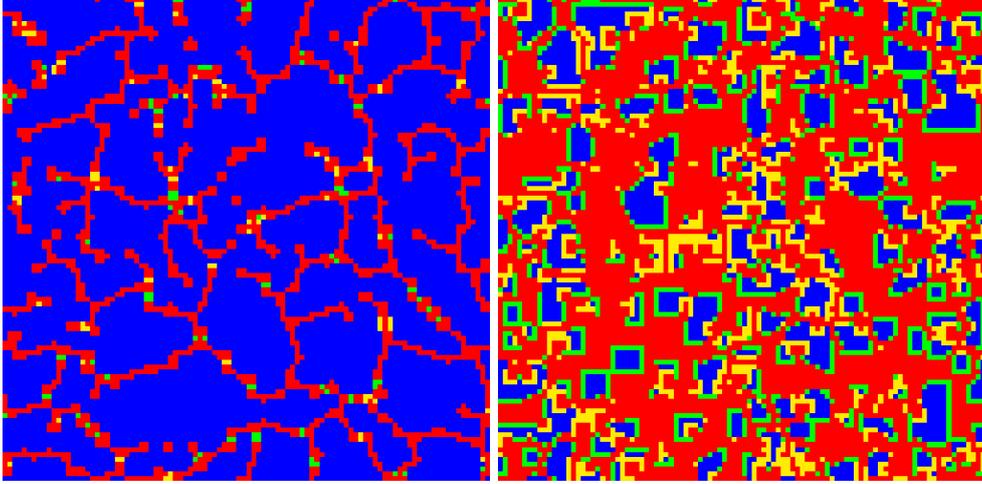
Figure 2. Snapshots of the game field for $b = 1.79$ (left) and $b = 1.81$ (right). The color coding is consistent with Ref. [9, 11]: blue is $\mathcal{C}$, red is $\mathcal{D}$, yellow is a $\mathcal{D}$ which was a $\mathcal{C}$ in the previous round, and green is a $\mathcal{C}$ which was a $\mathcal{D}$ in the previous round. See text for discussion.

Then we check the next site in the row $(0, 1)$, which is connected (as a neighbor to the site $(0,0)$) and assign it to the same cluster number 1. The same happens with the third site $(0,2)$. The fourth site $(0,3)$ is a $\mathcal{D}$, and we assign it to the cluster number 2, and so on. Figure 4 shows what happens when the row-wise scan of the lattice reaches the site $(6, 4)$. Its left-hand neighbor belongs to the cluster 1, and the upper site belongs to the cluster 3. Site $(6, 4)$ is assigned to the cluster 1, as well as all sites which were assigned to the cluster 3, are reassigned to the cluster 1. Process of reassignment is generated by the red arrow. After the decomposition is finished, we calculate cluster "masses" or "sizes" (i.e., the number of sites which belongs to a given cluster), and update the histograms of the cluster sizes.

To extract the interface between $\mathcal{C}$-clusters and $\mathcal{D}$-clusters, we use the procedure of ref. [20]. It is defined on the dual lattice, which is plotted through the middle of the edges connecting sites, and perpendicular to them. An interface is shown by a black line in the right panel of Figure 4. We extract the external interface of the percolating cluster (it is $\mathcal{C}$-cluster in the figure), and calculate the average length $\mathcal{L}$, which is the average length of the percolating interfaces in the steady-state regime. If the game field does not contain a percolating interface at this time step,
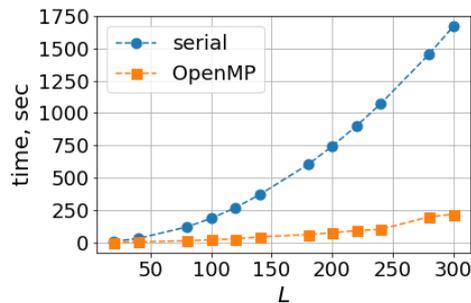
Figure 3. Computation time for a sequential and OpenMP implementations of the algorithm with 12 OpenMP threads. Each simulation is 50 independent realizations of initial conditions, simulated for 500 generations.
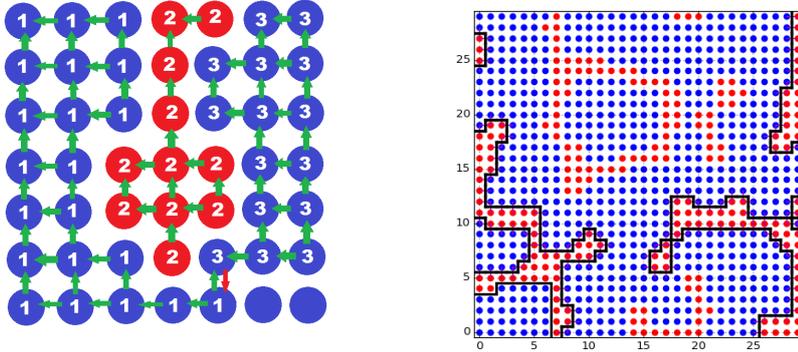
Figure 4. **Left**: An illustration of the Hoshen-Kopelman algorithm. See text for details. **Right**: An illustration on the interface extraction. (black lines) Color coding is the same as in the Figure 2: blue dots are $\mathcal{C}$ and red dots are $\mathcal{D}$.

we do not update the averages or histograms.

4. Results of simulations and data analysis

4.1. Phase transition

The average density of $\mathcal{C}$ changes drastically from $f_c = 0.737(6)$ at $b = 1.775$ to $f_c = 0.315(6)$ at $b = 1.825$ (see the left panel of Figure 1). We stress that the latter value is consistent with the magic density [11], $f_c = 12 \log 2 - 8 \approx 0.318$, which corresponds to the growth of $\mathcal{D}$ cluster. This way, we can attribute the transition from the $\mathcal{C}$-cluster percolation at $3/2 < b < 9/5$ (the blue one in the left panel of Figure 2) to the $\mathcal{D}$-cluster percolation (the red one in the right panel of Figure 2). In other words, the transition is from a random oscillatory regime of $\mathcal{C}$-clusters to the random growth regime of $\mathcal{D}$-clusters. The density changes abruptly at $b = b_c = 9/5$.

In both regimes, $b < 9/5$ and $b > 9/5$, the cluster size distribution demonstrates an exponential decay with size, although range of the distribution is much large in the case of proliferation of $\mathcal{D}$-clusters.

4.2. Scaling of the interface length

We fit the interface length $\mathcal{L}$ with a power law form,

$$\mathcal{L} = AL^\theta + c . \tag{1}$$

The result for $b = 1.74$ is $A = 0.22(4)$, $c = 5(9)$, and $\theta = \mathbf{2.07(5)}$. For $b = 1.81$ it is $A = 0.35(1)$, $c = -20(5)$, and $\theta = \mathbf{1.99(1)}$. Estimates of the values of c are rather rough, and we include then to extract some idea on the typical correlation length. What is very intriguing is that the interface of the percolating clusters in both regimes is close to $\mathbf{2}$, which is not obvious while looking in the Figures 2. Value of $\theta = 2$ corresponds to the line which is space-filling in the thermodynamic limit of $L \to \infty$.

4.3. Minkowski dimension of the interface

To check this finding we estimate the Minkowski dimension of the interface, i.e., we cover the interface with square boxes of linear size l, and count the minimum number of boxes $N(l)$ needed to completely cover the interface, and take the limit of $l \to 0$

$$d_M = \lim_{l \to 0} \frac{\log N(l)}{-\log l} . \tag{2}$$

This quantity is much more sensitive to the finite size of the lattice L. There are also visible deviations for $l \sim 1$, i.e., for the box size comparable to the lattice spacing. We fit data for $N(l)$ with the linear formula, $\log N(l) = B \log (1/l) + c$. Results of this fit procedure show a clear trend of d_M to approaching the value of 2 in the thermodynamic limit $L \to \infty$.

5. Discussion and future work

We study an evolutionary game based on the Prisoner's Dilemma with agents arranged on a square grid in the plane. Despite its apparent simplicity, this deterministic, globally synchronous game displays surprisingly rich behavior in the long time limit. The steady state of the game features a series of dynamic regimes separated by sharp transitions at specific values of the payoff parameter. Transitions are characterized by jumps of the average density of cooperators, and non-trivial rearrangements of the game field. Interfaces between clusters of players with different strategies are stochastic fractals, which, however, are space-filling across the transitions, thus resembling specific Julia sets which are also space-filling [21].

It would be interesting to further characterize these emergent geometric structures. It would also be interesting to investigate the game for more complicated arrangements of players, such as non-square grids, complex networks and dynamic graphs.

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