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Parametric Resonance in the Logistic Equation with Delay under a Two-Frequency Perturbation

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The logistic equation with delay feedback circuit and with periodic perturbation parameters is considered. Parameters of the problem (coefficient of linear growth and delay) are chosen close to the critical values at which cycle is bifurcated from equilibrium point. We assume that these values have double-frequency relation to the time and the frequency of action and doubled frequency of the natural vibration are close. Asymptotic analysis is performed under these assumptions and leads to a two-dimensional system of ordinary differential equations. Linear part of this system is periodic. If the parameter that defines frequency detuning of an external action is large or small we can apply standard asymptotic methods to the resulting system. Otherwise numerical analysis is performed. Using results of numerical analysis, we clarify the main scenarios of phase transformations and find the region of chaotic oscillations. It is main conclusion that in the case of parametric resonance the dynamics of the problem with double-frequency perturbation is more complicated than dynamics of the problem with single-frequency perturbation.

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1. Problem statement

The logistic equation with delay

$$\dot{u} = r[1 - u(t - T)]u \tag{1}$$

is a model for a wide range of processes and phenomena that depend on the certain conditions the system had at some point in the past. From the applied point of view, only the positive values of the function $u(t)$ in Eq. (1) make sense. The positive parameter r characterizes the exponential growth of $u(t)$ for small values of this quantity. In mathematical ecology the coefficient r is referred to as Malthusian coefficient. The term $[1 - u(t - T)]$ in Eq. (1) defines the saturation that occurs with a certain positive time delay T. If $rT \leq 37/24 \approx 1.51$ then all (positive) solutions of Eq. (1) tend to $u_0 \equiv 1$ as $t \to \infty$. The equilibrium u_0 is asymptotically stable if $rT \leq \pi/2 \approx 1.57$. Probably, all solutions of Eq. (1) tend to u_0 as $t \to \infty$ if $rT \leq \pi/2$. Some results in this direction may be found in [1, 2].

Consider the linearized on u_0 equation

$$\dot{u} = -ru(t - T) \tag{2}$$

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and the corresponding characteristic equation

$$\mu = -r\exp(-\mu T). \tag{3}$$

If $rT < \pi/2$, all roots of Eq. (3) have negative real parts and, thus, all solution of Eq. (1) from sufficiently small neighborhood of u_0 tend to u_0 as $t \to \infty$. If $rT > \pi/2$, then the equation Eq. (3) has a root with positive real part. Hence, the problem of dynamics of Eq. (1) in the neighborhood of the u_0 becomes nonlocal. For $r = r_0$, $T = T_0$ where $r_0T_0 = \pi/2$, characteristic equation Eq. (3) has a pair of purely imaginary roots

$$\mu_{1,2} = \pm i\pi(2T_0)^{-1}, \tag{4}$$

and all other roots have negative real parts.

Consider the case, when the values of parameters r and T are close to r_0 and T_0 respectively. The following theorem holds (see, e.g., [3–6]).

Theorem 1. *Suppose that*

$$r = r_0 + \varepsilon r_1, \quad T = T_0 + \varepsilon T_1 \quad \textit{and} \quad 0 < \varepsilon \ll 1.$$

Then Eq. (1) *has a local two-dimensional stable invariant manifold in some neighborhood of the equilibrium* u_0 *(in metric* $C[-T, 0]$*). The dynamics of Eq.* (1) *on this manifold is described, accurate within the terms of the order* $O(\varepsilon^{1/2})$*, by the scalar complex differential equation*

$$\frac{d\xi}{d\tau} = \alpha\xi + d|\xi|^2\xi \tag{5}$$

where $\tau = \varepsilon t$,

$$\alpha = (r_0^2T_1 + ir_1)(1 + i\pi/2)^{-1},$$
$$d = d_0 + ic_0 = -\frac{r_0(3\pi - 2 + i(\pi + 6))}{10(1 + \pi^2/4)}, \tag{6}$$

and $\operatorname{Re} d < 0$. *Function* $\xi(\tau)$ *is connected with solution of Eq.* (1) *by the relation*

$$u(t,\varepsilon) = 1 + \varepsilon^{1/2}\Big[\xi(\varepsilon t)\exp\big(i\pi(2T_0)^{-1}t\big) + \bar{\xi}(\varepsilon t)\exp\big(-i\pi(2T_0)^{-1}t\big)\Big] + \varepsilon u_2(t,\tau) + \varepsilon^{3/2}u_3(t,\tau) + O(\varepsilon^2) \tag{7}$$

where $u_j(t,\tau)$, $j = 2, 3$ *are* $4T_0$*-periodic in* t *functions. Consequently, the stable cycle of Eq.* (5) *corresponds to a certain stable periodic solution of Eq.* (1) *that is described by the formula* (7).

In this paper we study the local (in the neighborhood of the equilibrium u_0) dynamics of the logistic equation with delay Eq. (1) under a two-frequency perturbation. We assume that coefficients r and T in Eq. (1) depend on time and have the following form:

$$r = r_0 + \varepsilon r_1 + \varepsilon r_{11}\sin\omega_1 t + \varepsilon r_{12}\sin\omega_2 t, \tag{8}$$
$$T = T_0 + \varepsilon T_1 + \varepsilon T_{11}\sin\omega_1 t + \varepsilon T_{12}\sin\omega_2 t \tag{9}$$

where r_1 and T_1 are nonpositive and, moreover,

$$r_1 + T_1 < 0. \tag{10}$$

Inequality Eq. (10) means that the equilibrium u_0 in Eq. (1) (and $\xi \equiv 0$ in Eq. (5)) is asymptotically stable if there are no perturbations with zero mean value of the form Eq. (8), (9). It is not difficult to show that if $\omega_{1,2} = \pi/T_0$ then for sufficiently small values of the parameter $\varepsilon > 0$ all solutions of Eq. (1) from some sufficiently small neighborhood of u_0 (that does not depend on ε) tend to u_0 as $t \to \infty$.

Here we study the parametric resonance phenomenon under a two-frequency perturbation, i.e., we assume that the frequencies ω_1 and ω_2 are close to the double frequency π/T_0 of the natural oscillations in Eq. (2):

$$\omega_1 = \pi/T_0 + \varepsilon\delta_1, \quad \omega_2 = \pi/T_0 + \varepsilon\delta_2. \tag{11}$$

We will show in this situation the dynamics of Eq. (1) is rather interesting. We emphasize that the case of two-frequency perturbation is much more complicated than the case of one-frequency perturbation, when $\omega_1 = \omega_2$. We note that in linear approximation the problem of the parametric resonance under two-frequency perturbations was discussed in [7]. Some applied aspects of the system dynamics of the laser models with an oscillating delay were considered in [8].

2. Algorithmic part

The proposed algorithm is based on the formal relation Eq. (7). By the use of Eq. (7) we reduce Eq. (1) under conditions Eq. (8), (9) to scalar ordinary differential complex equation for slowly varying amplitude $\xi(\tau)$.

We substitute Eq. (7) in Eq. (1) and collect terms with like powers of ε. On the third step, collecting terms with $\varepsilon^{3/2}$, we get the following equation for $\xi(\tau)$ from the solvability condition of the equation for $u_3(t,\tau)$:

$$\frac{d\xi}{d\tau} = \alpha\xi + A(\tau)\bar{\xi} + d|\xi|^2\xi \qquad (12)$$

where $A(\tau) = \alpha_1 \exp(i\delta_1\tau) + \alpha_2 \exp(i\delta_2\tau)$ and $\alpha_1 = T_{11}r_0^2(2i-\pi)^{-1}$, $\alpha_2 = T_{12}r_0^2(2i-\pi)^{-1}$. Introducing in Eq. (12) the changes $\xi = v\exp(i\delta_1\tau/2)$, $\delta_2 - \delta_1 = \mu$ $(\delta_2 > \delta_1)$ and $\mu\tau = \tau_1$ we obtain the following equation with 2π-periodic coefficients:

$$\mu\frac{dv}{d\tau_1} = \alpha_0 v + (\alpha_1 + \alpha_2\exp(i\tau_1))\bar{v} + d|v|^2 v. \qquad (13)$$

Here $\alpha_0 = \alpha - i\delta_1$ and we recall also that $\operatorname{Re}\alpha_0 < 0$, $\operatorname{Re} d < 0$.

Let us study the dynamics of Eq. (13) for various values of the parameters μ, δ_1, α_1 и α_2. Note, that, by the formula (7) the coarse periodic solution of Eq. (13) corresponds to the two-dimensional torus of the same stability.

The equation (13) is well-studied in the case of the parametric resonance under one-frequency perturbation, i.e., when $\alpha_2 = 0$ (see, for instance, [9]).

In [7] the stability analysis for the linear part of Eq. (13) was conducted. We describe here some of the obtained results. First we rewrite the linear part of Eq. (13) in the real form

$$\mu\frac{dw}{d\tau_1} = B(\tau_1)w,$$
$$w = \begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix},$$
$$B(\tau_1) = \begin{pmatrix} \operatorname{Re}\alpha_0 + B_1(\tau_1) & -\operatorname{Im}\alpha_0 + B_2(\tau_1) \\ \operatorname{Im}\alpha_0 + B_2(\tau_1) & \operatorname{Re}\alpha_0 - B_1(\tau_1) \end{pmatrix} \qquad (14)$$

where $B_1(\tau_1) = \operatorname{Re}\alpha_1 + \operatorname{Re}\alpha_2\cos\tau_1 - \operatorname{Im}\alpha_2\sin\tau_1$, $B_2(\tau_1) = \operatorname{Im}\alpha_1 + \operatorname{Re}\alpha_2\sin\tau_1 + \operatorname{Im}\alpha_2\cos\tau_1$.

For sufficiently large μ we can apply the well-known method of averaging [13]. The stability properties are defined by the averaged system

$$\mu\frac{dw}{d\tau_1} = B_0 w,$$
$$B_0 = \begin{pmatrix} \operatorname{Re}\alpha_0 + \operatorname{Re}\alpha_1 & -\operatorname{Im}\alpha_0 + \operatorname{Im}\alpha_1 \\ \operatorname{Im}\alpha_0 + \operatorname{Im}\alpha_1 & \operatorname{Re}\alpha_0 - \operatorname{Re}\alpha_1 \end{pmatrix}.$$

Thus, for sufficiently large μ we can reduce the studied case to the problem of the parametric resonance under one-frequency perturbation.

Suppose now that

$$0 < \mu \ll 1.$$

It follows from the results in [7, 10] that the stability of zero solution of Eq. (14) is defined by the function

$$\rho(\tau_1) = \det\big(B(\tau_1) - \operatorname{Re}\alpha_0 \cdot E\big). \qquad (15)$$

The following three cases should be considered depending on the values the function $\rho(\tau_1)$ may take: only positive, only negative or it may change its sign. Taking into account the formula for the matrix $B(\tau_1)$ in Eq. (14) we obtain

$$\rho(\tau_1) = (\operatorname{Im}\alpha_0)^2 - (B_1^2(\tau_1) + B_2^2(\tau_1)). \qquad (16)$$

Thus, by varying $\operatorname{Im}\alpha_0$, we may produce each of the mentioned above cases. Acting in the same manner as in [7], we can prove the following theorem that describes the dynamics of the system (14).

Theorem 2. *The following cases are possible for the system* (14).

1. Suppose that $\rho(\tau) > 0$ *for all* $\tau > 0$. *Then there exists* $\mu_0 > 0$ *such that for* $0 < \mu \leq \mu_0$ *zero solution of Eq.* (14) *is asymptotically stable.*

2. Suppose that $\rho(\tau) < 0$ *for all* $\tau > 0$. *Then there exists* $\mu_0 > 0$ *such that for* $0 < \mu \leq \mu_0$ *zero solution of Eq.* (14) *is asymptotically stable (unstable), if*

$$\frac{1}{2\pi}\int_0^{2\pi}\sqrt{-\rho(\tau)}d\tau + \operatorname{Re}\alpha_0 < 0\ (> 0). \qquad (17)$$

3. If function $\rho(\tau)$ *changes its sign then the system* (14) *is a system with turning points. Besides, if inequality*

$$\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{|\rho(\tau)| - \rho(\tau)}{2}}d\tau > \operatorname{Re}\alpha_0, \qquad (18)$$

holds, then stability and instability intervals of Eq. (14) *alternate infinitely as* $\mu \to 0$. *(We note also, that if the sign in Eq.* (18) *is "<" then zero solution is asymptotically stable for all sufficiently small* μ.*)*

The mentioned properties of the system (14) allows us to make conclusions concerning the local stability or instability of zero solution of nonlinear Eq. (13) for sufficiently large or sufficiently small μ. Since we have no methods to study the dynamics of Eq. (13) for the values of parameter μ between «sufficiently small» and «sufficiently large» we apply numerical methods of analysis in this case. We can use these methods for investigation of the stable regimes of Eq. (13) due to its dissipativity ($\operatorname{Re} d < 0$). The latter makes it possible to choose initial conditions only from some neighborhood of zero on the phase plane.

3. Numerical analysis results

We turn to numerical analysis of the nonlinear equation (13). By analogy with Eq. (14) from the equation (13) we have

$$\begin{aligned} w_1' = \nu\big((\operatorname{Re}\alpha_0 + B_1(\tau_1))w_1 - (\operatorname{Im}\alpha_0 - \\ -B_2(\tau_1))w_2 + (d_0 w_1 - c_0 w_2)(w_1^2 + w_2^2)\big), \\ w_2' = \nu\big((\operatorname{Im}\alpha_0 + B_2(\tau_1))w_1 + (\operatorname{Re}\alpha_0 \\ -B_1(\tau_1))w_2 + (c_0 w_1 + d_0 w_2)(w_1^2 + w_2^2)\big) \end{aligned} \qquad (19)$$

where $\nu = 1/\mu$, $d = d_0 + ic_0$, and w_1, w_2 are the components of the vector w. Note that, along with being dissipative, the system (19) is symmetric with respect to simultaneous change of sign of the variables w_1 and w_2.

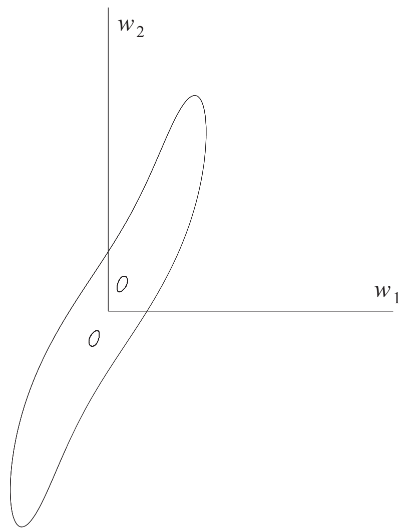

FIG. 1. Phase portrait of the system (19), obtained for $\nu = 0.4$.

If the matrix $B(\tau_1)$ does not depend on time, the system (19) undergoes phase portrait changes in a usual way described, for example, in [9] or [14]. Only equilibria and cycles can be stable in the system (19) in this case (One more complicated case was discussed in [15]).

In presence of periodic forcing dynamics of the system (19) becomes much more complicated possibly exhibiting irregular oscillations. Typical situations emerging in this case are described, e.g., in [14]. System Eq. (19) depends on a large number of parameters whose variation leads to different phase portrait changes. Let us describe in details the situation when chaotic oscillations appear.

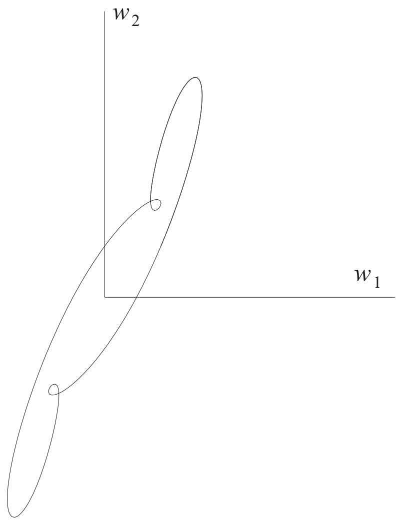

FIG. 2. Phase portrait of the system (19), obtained for $\nu = 0.8$.

Suppose the problem parameters are fixed as follows:

$$r_1 = 0,\ T_1 = -0.1,\ \delta_1 = 1,\\ T_{11} = 1,\ T_{12} = 0.1, \tag{20}$$

parameters d_0, c_0 are defined by Eq. (6) to be $d_0 \approx -0.5283$, $c_0 \approx -0.6505$. Taking ν as a bifurcation parameter we increase its value. In this situation the dynamics of the system (19) is defined by the following phase portrait changes determined numerically. Only fragments of the dynamics are shown in the areas with chaotic behavior.

1. For sufficiently small $\nu < \nu_1$, $\nu_1 \approx 0.369$ the system (19) possesses two symmetric stable cycles C_1, C_1^*.

2. At $\nu = \nu_1$ a stable self-symmetric cycle C^S is born, which for $\nu_1 < \nu < \nu_2$, $\nu_2 \approx 0.435$ coexists with cycles C_1, C_1^* found earlier (on Fig. 1 the phase portrait of the system (19) is depicted for $\nu = 0.4$ showing two symmetric cycles C_1, C_1^* and self-symmetric C^S).

3. At $\nu = \nu_2$ the cycles C_1, C_1^* disappear (merge with the zero equilibrium) and for $\nu_2 < \nu < \nu_3$, $\nu_3 \approx 0.895$ only the cycle C^S exists (see Fig. 2, obtained for $\nu = 0.8$).

4. At $\nu = \nu_3$ self-symmetric cycle C^S loses its symmetry and splits into two cycles C_2, C_2^* symmetric to each other which coexist for $\nu_3 < \nu < \nu_4$, $\nu_4 \approx 0.933$.

5. $\nu = \nu_4$ is a period doubling bifurcation point for C_2, C_2^* (see Fig. 3, obtained for $\nu = 0.94$).

6. On the interval $\nu_4 < \nu < \nu_5$, $\nu_5 \approx 0.945$ both cycles C_2, C_2^* undergo a cascade of period doubling bifurcations leading to emergence of two chaotic regimes symmetric to each other (on Fig. 4 the phase portrait of one of the chaotic regimes of the system (19) is depicted for $\nu = 0.95$).

7. For $\nu_5 < \nu < \nu_6$, $\nu_6 \approx 0.961$ we observe two chaotic regimes symmetric to each other.

8. On the interval $\nu_6 < \nu < \nu_7$, $\nu_7 \approx 0.966$ the reverse cascade of period doubling bifurcations takes place leading to a pair of symmetric cycles C_3, C_3^*.

9. On the interval $\nu_7 < \nu < \nu_8$, $\nu_8 \approx 0.97$ symmetric cycles C_3, C_3^* are stable, and at $\nu = \nu_8$ via the separatrix splitting bifurcation self-symmetric chaos emerges (on Fig. 5 the phase portrait of self-symmetric chaotic regime of the system (19) is depicted for $\nu = 1$).

10. At $\nu = \nu_9$, $\nu_9 \approx 1.015$ via the inverse separatrix splitting bifurcation two cycles C_4, C_4^* symmetric to each other appear that exist for $\nu > \nu_9$ (on Fig. 6 the phase portrait of one of the two stable cycles of the system (19) is depicted for $\nu = 1.05$).

Phase portrait changes that the system (19) undergoes in the area of irregular oscillations are visualized using the plot of the dependence of the maximal Lyapunov exponent of the system (19) on the parameter ν, represented in Fig. 7. Computations were made with parameter step $\Delta\nu = 0.5 \cdot 10^{-4}$. Two more plots of dependence of the maximal Lyapunov exponent of the attractor are presented in Fig. 8, 9. They are computed for the same parameter values Eq. (20) except for the parameter T_{12} and parameter ν step, that are taken to be $T_{12} = 0.9$, $\Delta\nu = 0.5 \cdot 10^{-4}$ for Fig. 8 and $T_{12} = 0.223$, $\Delta\nu = 0.25 \cdot 10^{-4}$ for Fig. 9. The plot $\lambda_{\max}(\nu)$ is evidently nonsmooth, and, moreover, there is large, possibly unbounded number of points where $\lambda_{\max}(\nu)$ crosses the x-

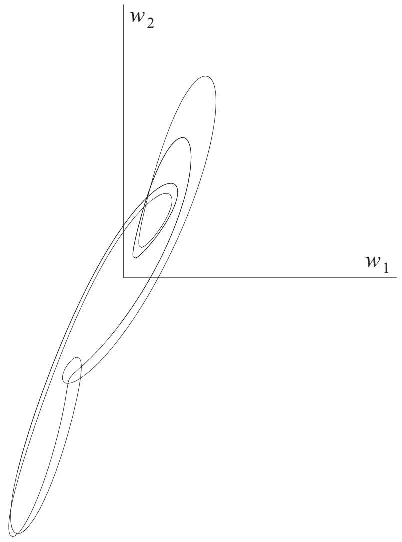

FIG. 3. Phase portrait of the system (19) obtained for $\nu = 0.94$.

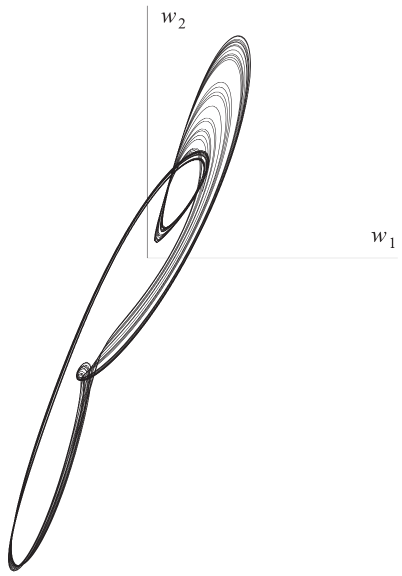

FIG. 4. Phase portrait of system (19) obtained for $\nu = 0.95$

axis. We also want to pay attention to the points where the function $\lambda_{\max}(\nu)$ vanishes remaining nonnegative. These are the points where period doubling or symmetry breaking bifurcations of the cycles of the system (19) occur.

Lyapunov exponents were computed by the

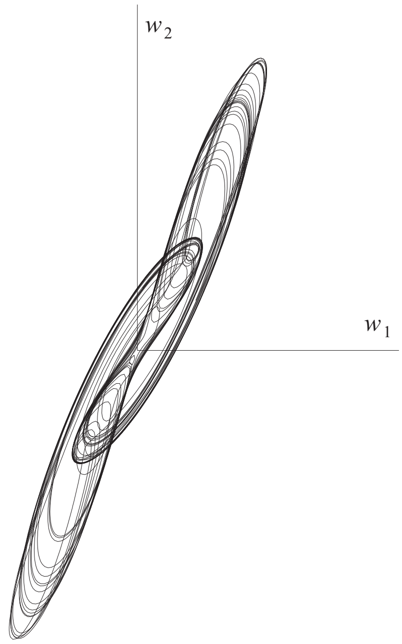

FIG. 5. Phase portrait of the system (19) obtained for $\nu = 1.0$.

package Tracer3.70, based on a variation of the Bennetin algorithm proposed in [16].

4. Final remarks

The results above can be expanded (see [11, 12]) to the parabolic equation

$$\frac{\partial u}{\partial t} = \varepsilon\gamma\frac{\partial^2 u}{\partial x^2} + r[1 - u(t-T)]u \tag{21}$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t, x). \tag{22}$$

Using Eq. (8), (9) we obtain a boundary problem to determine slowly changing amplitudes $\xi(\tau, x)$:

$$\frac{\partial \xi}{\partial \tau} = \gamma\Big(1+i\frac{\pi}{2}\Big)^{-1}\frac{\partial^2 \xi}{\partial x^2}+\alpha\xi+A(\tau)\bar{\xi}+d|\xi|^2\xi, \tag{23}$$

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x). \tag{24}$$

The relation between solutions of this problem and solutions of Eq. (21), (22) is defined by

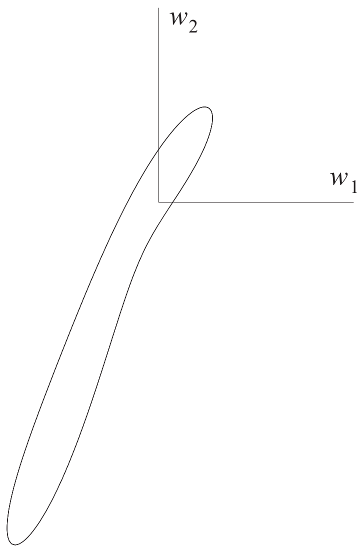

FIG. 6. Phase portrait of the system (19) obtained for $\nu = 1.05$.

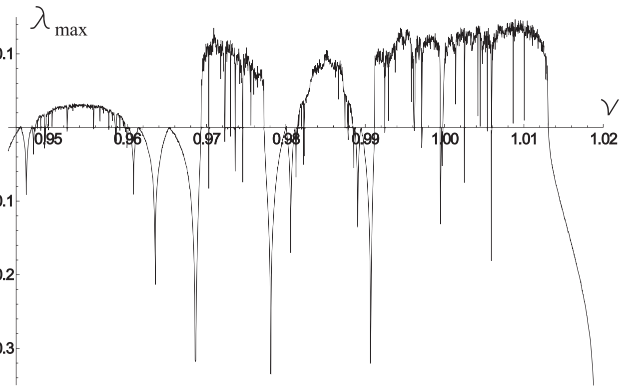

FIG. 7. Dependence of maximal Lyapunov exponent $\lambda_{\max}(\nu)$ for $T_{12} = 0.1$, $\Delta\nu = 0.25 \cdot 10^{-4}$.

the asymptotic formula (7) where $\xi(\tau)$ should be replaced by $\xi(\tau, x)$. Dynamics of Eq. (23, 24) can be significantly more rich and diverse comparing with Eq. (12).

Main results.

1. Dynamics in the case of parametric resonance under two-frequency perturbation is dramatically more complex comparing with the one-frequency perturbation case. There appear cycles (see Fig. 2,3) and regions (in the

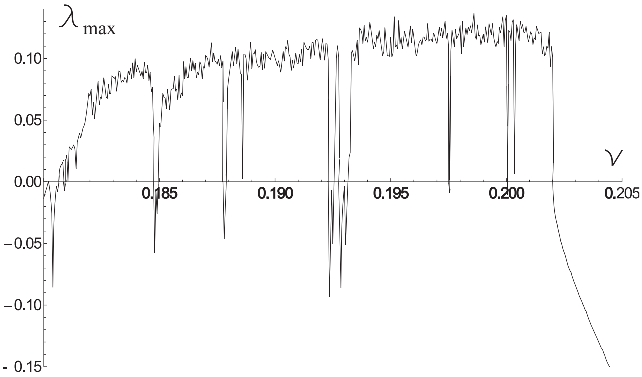

FIG. 8. Dependence of maximal Lyapunov exponent $\lambda_{\max}(\nu)$ for $T_{12} = 0.9$, $\Delta\nu = 0.5 \cdot 10^{-4}$.

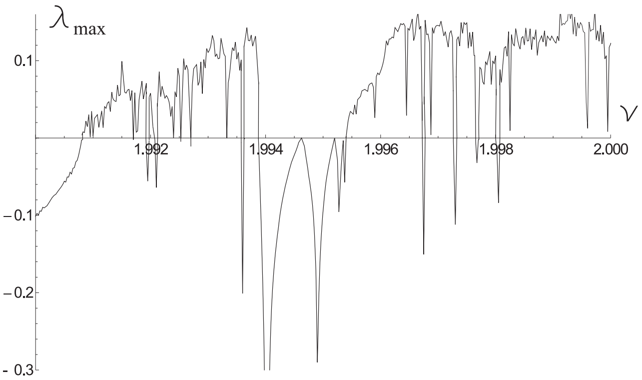

FIG. 9. Dependence of maximal Lyapunov exponent $\lambda_{\max}(\nu)$ for $T_{12} = 0.223$, $\Delta\nu = 0.25 \cdot 10^{-4}$.

parameter space) of irregular solution behavior of significantly more complex form.

2. In the case of $\rho(\tau_1) < 0$, i. e., when zero solution is asymptotically stable, no nonstationary stable solutions of Eq. (1) were found.

3. For sufficiently large μ the dynamical properties of Eq. (1) are relatively simple. For sufficiently small μ we have complex stable relaxation cycles. Either they are symmetric or there exist two stable cycles that are symmetric to each other.

4. Irregular oscillations are specific for «intermediate» values of the parameter μ.

5 Irregular oscillations of the system (19) are realized in relatively narrow intervals of ν. Phase changes in this case are due to symmetry breaking bifurcations, period doubling bifurcation cascades and separatrix splitting bifurcation.

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