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Blue Sky Catastrophe as Applied to Modeling of Cardiac Rhythms

S. D. Glyzin[a], A. Yu. Kolesov[a], and N. Kh. Rozov[b]

[a] *Faculty of Mathematics, Yaroslavl State University, ul. Sovetskaya 14, Yaroslavl, 150000 Russia*
e-mail: glyzin@uniyar.ac.ru, andkolesov@mail.ru
[b] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia*
e-mail: fpo.mgu@mail.ru

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Abstract—A new mathematical model for the electrical activity of the heart is proposed. The model represents a special singularly perturbed three-dimensional system of ordinary differential equations with one fast and two slow variables. A characteristic feature of the system is that its solution performs nonclassical relaxation oscillations and simultaneously undergoes a blue sky catastrophe bifurcation. Both these factors make it possible to achieve a phenomenological proximity between the time dependence of the fast component in the model and an ECG of the human heart.

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1. FORMULATION OF THE PROBLEM

First, we say a few words about the blue sky catastrophe. This term is used to refer to a nonlocal bifurcation of codimension 1, which, in the simplest case, can be described as follows.

Consider a smooth one-parameter family of vector fields X_μ in $\mathbb{R}^3$. Assume that, at $\mu = 0$, the flow X_μ has a periodic trajectory L_0 of simple saddle-node type. Consider a sufficiently small neighborhood $\mathcal{U}$ of L_0 that is partitioned by a two-dimensional strongly stable manifold $W^{ss}(L_0)$ into two domains: a node one $\mathcal{U}^+$, all trajectories from which tend to L_0 as $t \longrightarrow +\infty$ and a saddle domain $\mathcal{U}^-$ containing a two-dimensional unstable manifold $W^u_{\text{loc}}(L_0)$ with boundary L_0. The following constraint is substantially nonlocal in character: as t increases, all trajectories of the system X_0 with initial values from $W^u_{\text{loc}}(L_0)$ leave $\mathcal{U}$ and then return to it, hitting the node domain $\mathcal{U}^+$. Then, obviously, each of these trajectories is doubly asymptotic to L_0. Finally, we assume that the set $W^u(L_0)$ obtained from $W^u_{\text{loc}}(L_0)$ by extension along the trajectories of X_0 is not a topological manifold (in the three-dimensional case, this means that its closure is not homeomorphic to a two-dimensional torus).

According to [1], under the constraints formulated above and some additional technical conditions, the vanishing of the saddle-node cycle L_0 in the system X_μ, $0 < \mu \leq 1$, leads to the appearance of a stable closed trajectory $L(\mu)$ whose period and length tend to infinity as $\mu \longrightarrow 0$. The upper topological limit of $L(\mu)$ as $\mu \longrightarrow 0$ is the set $W^u(L_0) \cup L_0$. This bifurcation is known as the blue sky catastrophe.

The feasibility of this bifurcation in singularly perturbed systems with one slow and m fast variables (where $m \geq 2$) was illustrated in [2, 3]. In [4] a blue sky bifurcation was studied in the system

$$\dot{x} = f(x, y, \mu), \quad \varepsilon\dot{y} = g(x, y), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad y \in \mathbb{R}, \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $|\mu| \ll 1$, and the functions $f, g \in C^\infty$ satisfy the standard constraints (see [5]), which ensure the existence of classical relaxation oscillations. Recall that oscillations whose slow components x_1 and x_2 tend, as $\varepsilon \longrightarrow 0$, to continuous functions of t, while the fast component y converges pointwise to a discontinuous function are called *classical*.

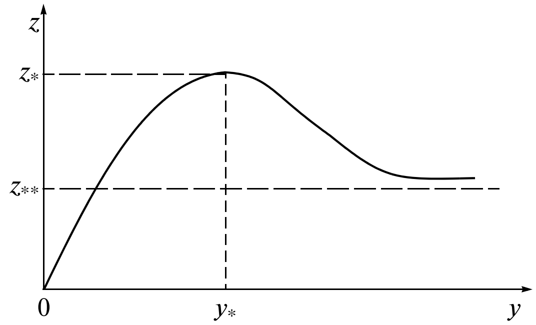

Fig. 1.

In this paper, the results of [4] are extended to a three-dimensional system similar to (1.1), namely,

$$\dot{x} = f_1(x, \mu) + \sqrt{\varepsilon}y^2 f_2(x), \quad \varepsilon\dot{y} = g(x) - h(y); \tag{1.2}$$

here, as before, $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$, $0 < \varepsilon \ll 1$, and $|\mu| \le \mu_0$, where $\mu_0 > 0$ is a sufficiently small constant. The functions $f_1(x, \mu) \in C^\infty(\mathbb{R}^2 \times [-\mu_0, \mu_0]; \mathbb{R}^2)$, $f_2(x) \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$, $g(x) \in C^\infty(\mathbb{R}^2)$, and $h(y) \in C^\infty(\mathbb{R})$ are assumed to satisfy special conditions that guarantee the existence of nonclassical relaxation oscillations. According to the terminology used in [6], nonclassical relaxation oscillations are ones such that, as $\varepsilon \longrightarrow 0$, their slow components converge pointwise to discontinuous functions, while their fast component is δ-like.

Let us describe in detail the constraints imposed on the right-hand side of system (1.2). The first two of them are related to the topology of the surface of slow motions

$$\Gamma = \{(x, y) : g(x) - h(y) = 0\}. \tag{1.3}$$

Condition 1. Assume that, first, $g(x) > 0$ $\forall x \neq 0$, $g(0) = 0$; second, for any fixed $z > 0$, the equation $g(x) = z$ defines a closed C^∞ in $\mathbb{R}^2$ that is homeomorphic to a circle.

Condition 2. There is $y = y_* > 0$ such that $h'(y) > 0$ for $y < y_*$, $h'(y) < 0$ for $y > y_*$, $h'(y_*) = 0$, $h''(y_*) < 0$, and $h(y_*) = z_* > 0$. Assume also that $h(0) = 0$ and, as $y \longrightarrow +\infty$, we have the asymptotic equality

$$h(y) = z_{**} + \sum_{k=1}^{\infty} \frac{z_k}{y^k}, \quad z_{**} \in (0, z_*), \tag{1.4}$$

which remains valid after being differentiated with respect to y any number of times.

A visual representation of the properties of $z = h(y)$ is given by its plot presented in Fig. 1. By Conditions 1 and 2, surface (1.3) has the shape shown in Fig. 2.

Indeed, let $y = y_1(z)$ and $y = y_2(z)$ denote the roots of the equations $h(y) = z$ for $z \in (-\infty, z_*]$ and $h(y) = z$ for $z \in (z_{**}, z_*]$ on the intervals $(-\infty, y_*]$ and $[y_*, +\infty)$, respectively (the existence of these roots follows from Condition 2). Relying on Condition 1, we conclude that the surface Γ is partitioned into two parts: stable Γ_- (on which $h'(y) > 0$) and unstable Γ_+ (on which $h'(y) < 0$), which are separated by a falling line Γ_0 (where $h'(y) = 0$). These parts are defined as

$$\Gamma_- = \{(x, y) : y = \Phi_-(x), x \in \Omega_1\}, \quad \Gamma_+ = \{(x, y) : y = \Phi_+(x), x \in \Omega_2\}, \\ \Gamma_0 = \{(x, y) : y = y_*, x \in l_1\}, \tag{1.5}$$

$$\Phi_-(x) = y_1(z)|_{z = g(x)}, \quad \Phi_+(x) = y_2(z)|_{z = g(x)}. \tag{1.6}$$

In (1.5) $\Omega_1 \subset \mathbb{R}^2$ denotes an internal domain bounded by the simple closed curve $l_1 = \{x \in \mathbb{R}^2$ is: $g(x) = z_*\}$ and $\Omega_2 \subset \mathbb{R}^2$ is an annular domain bounded by l_1 and the simple closed curve $l_2 = \{x \in \mathbb{R}^2 : g(x) = z_{**}\} \subset \Omega_1$. Note also that, since $\lim \Phi_+(x) = +\infty$ for $x \in \Omega_2$, $x \longrightarrow l_2$, the surface Γ represents a “bottle with an infinitely long neck” (see Fig. 2).

Now, we describe the constraints imposed on the vector function $f_2(x)$ in (1.2). Consider the auxiliary system

$$\dot{x} = f_2(x). \tag{1.7}$$

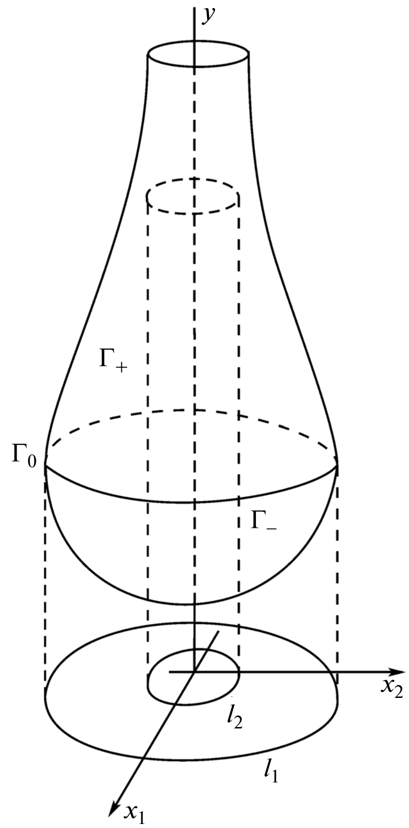

Fig. 2.

Condition 3. All trajectories of system (1.7) with initial values lying on the curve l_1 reach l_2 in a finite time (see Fig. 3). These trajectories have no contacts with the curve l_2, i.e.,

$$(\mathrm{grad}\, g(x), f_2(x)) < 0 \quad \text{for } x \in l_2, \tag{1.8}$$

where (*,*) is the Euclidean inner product.

The final series of constraints is concerned with the system

$$\dot{x} = f_1(x, \mu), \quad x \in \Omega_1 \cup l_1. \tag{1.9}$$

Before formulating the corresponding conditions, we consider a new curve l_3 lying in the domain bounded by l_2 (see Fig. 3). Accordingly, we need some additional constructions.

Fix an arbitrary point $x = x_0$ lying in a sufficiently small neighborhood of l_1. Let $x(t, x_0)$, $x(0, x_0) = x_0$ denote the solution of the Cauchy problem $\dot{x} = f_2(x)$, $x|_{t=0} = x_0$. Define the function

$$a(t, x_0) = \int_0^t [g(x(\theta, x_0)) - z_{**}]d\theta. \tag{1.10}$$

It has the properties

$$a(0, x_0) = 0, \quad \frac{\partial a}{\partial t}(t, x_0) = g(x(t, x_0)) - z_{**} > 0 \quad \text{for } 0 \le t < t_*,$$
$$\frac{\partial a}{\partial t}(t, x_0) < 0 \quad \text{for } t > t_*, \quad \lim_{t \to +\infty} a(t, x_0) = -\infty, \tag{1.11}$$

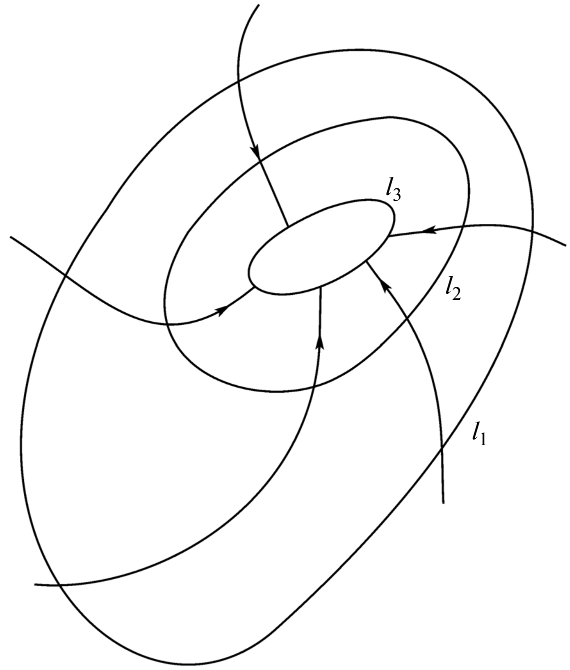

Fig. 3.

where $t_* = t_*(x_0)$ is the root of the equation

$$g(x(t, x_0)) - z_{**} = 0. \tag{1.12}$$

Indeed, assume that the point x_0 is so close to l_1 that $g(x_0) - z_{**} > 0$ and Condition 3 holds for the trajectory $x = x(t, x_0)$ of system (1.7). Then, over time, this trajectory exactly once intersects the curve l_2 and, according to inequality (1.8), as t increases further, it stays in the domain $\{x \in \mathbb{R}^2 : g(x) - z_{**} < 0\}$ without approaching its boundary. In turn, it follows that, first, Eq. (1.12) has a unique root $t = t_*(x_0) > 0$ with required properties (see (1.11)); and, second, for all sufficiently large t, the integrand in (1.10) is negative and bounded away from zero. Therefore, as $t \longrightarrow +\infty$, the limiting equality in (1.11) is surely satisfied.

Relying on properties (1.11), it is easy to see that the equation $a(t, x_0) = 0$ has a unique solution $t = t_{**}(x_0)$ on the half-line $t > 0$. In view this circumstance, we consider the two-dimensional mapping

$$\Pi_0 : x_0 \longrightarrow x(t, x_0)\big|_{t = t_{**}(x_0)}. \tag{1.13}$$

Simple verification shows that this is a diffeomorphism defined in a sufficiently small neighborhood of l_1. This means that $l_3 = \Pi_0(l_1)$ is a simple closed curve of the class C^∞ (see Fig. 3). This curve is the desired one.

Returning to system (1.9), we assume that the following constraints hold.

Condition 4. System (1.9) at $\mu = 0$ has the phase portrait shown in Fig. 4. Specifically, in the domain Ω_1, there exists a unique equilibrium $x = \bar{x}$ that is an exponentially unstable node or focus and there exists a unique semistable cycle of simple saddle-node type

$$L_0 : x = x_0(\varphi), \quad \frac{d\varphi}{dt} = \omega_0, \quad x_0(\varphi + 2\pi) \equiv x_0(\varphi), \quad \omega_0 > 0, \tag{1.14}$$

which encloses the equilibrium. In a finite time, all trajectories from the annular domain bounded by l_1 and L_0 reach l_1, intersecting it in the generic case, i.e.,

$$(\mathrm{grad}\, g(x), f_1(x, 0)) > 0 \quad \forall x \in l_1. \tag{1.15}$$

Finally, the curve l_3 lies in the domain bounded by the cycle L_0, but does not contain or enclose the point $x = \bar{x}$ (see Fig. 4).

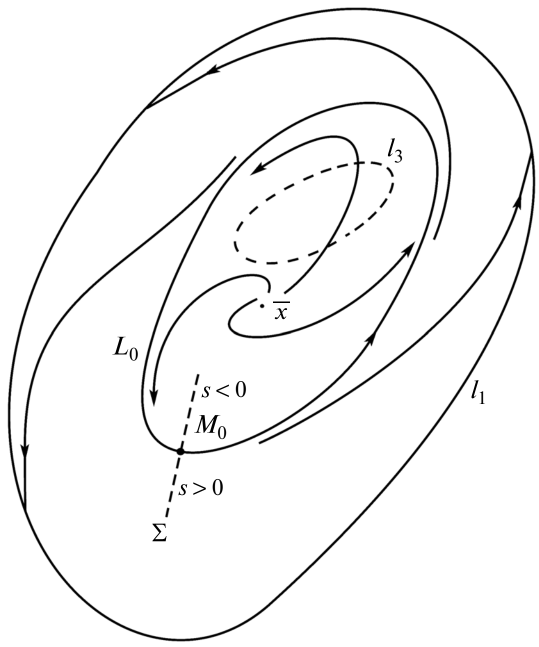

Fig. 4.

To explain the assumption made in Condition 4 that the saddle-node cycle (1.14) is simple, we fix an arbitrary point $M_0 \in L_0$ and denote by Σ a sufficiently short segment of the normal to L_0 at this point (see Fig. 4). The parameter s on the secant Σ is defined as the distance from an arbitrary point $M \in \Sigma$ to M_0 taken with a plus sign if M lies on the outer part of the normal and with a minus sign otherwise. Then, obviously, on the set $\{(s, \mu) : |s| \le s_0, |\mu| \le \mu_0\}$, where s_0, $\mu_0 > 0$ are suitably small, the Poincaré first return map

$$s \longrightarrow R(s, \mu), \quad R \in C^\infty([-s_0, s_0] \times [-\mu_0, \mu_0]) \tag{1.16}$$

is defined on trajectories of system (1.9); moreover, $R(s, 0)$ can be represented at the point $s = 0$ as a Taylor expansion of the form

$$R(s, 0) = s + d_0 s^2 + O(s^3). \tag{1.17}$$

The simplicity of the saddle-node cycle L_0 means that $d_0 > 0$.

Condition 5. Assume that

$$\alpha_0 = R'_\mu(0, 0) > 0, \tag{1.18}$$

where $R(s, \mu)$ is the function from (1.16).

It follows from properties (1.17) and (1.18) and the inequality $d_0 > 0$ that, for $\mu \in [-\mu_0, 0)$, map (1.16) has two fixed points $s_\pm(\mu) = \pm\sqrt{-\alpha_0\mu/d_0} + O(\mu)$, which are associated with two cycles in system (1.9): stable $L_-(\mu)$ and unstable $L_+(\mu)$, $L_\pm(0) = L_0$. Thus, in this case, all trajectories of the considered system with initial values lying on the curve l_3 tend to the cycle $L_-(\mu)$ as $t \longrightarrow +\infty$. If $\mu \in (0, \mu_0]$, then there are no cycles in system (1.9) and any of its trajectories with an initial value from l_3 first rotates asymptotically long (over time of order $1/\sqrt{\mu}$) in a neighborhood of the vanished cycle L_0 and then hits the curve l_1.

Under the above constraints, according to the results of [6], the qualitative behavior of any trajectory

$$(x(t, \varepsilon, \mu), y(t, \varepsilon, \mu)) : x(0, \varepsilon, \mu) = x_0, \quad y(0, \varepsilon, \mu) = y_0 \tag{1.19}$$

of system (1.2) with initial values x_0 and y_0 independent of ε or μ can be described as follows.

Suppose that the point (x_0, y_0) belongs to a sufficiently small neighborhood of the stable slow manifold Γ_- and x_0 lies in the annular domain bounded by L_0 and l_1. Then, after an asymptotically short time (on the order of $\varepsilon\ln(l/\varepsilon)$), phase point (1.19) "falls" on the surface Γ_- roughly along the line $x = x_0$ and then the motion continues in an asymptotically small (in ε) neighborhood of the curve $(x_1(t, \mu), y_1(t, \mu))$, where $y_1(t, \mu) = \Phi_-(x_1(t, \mu))$, $\Phi_-(x)$ is the function from (1.6), and $x_1(t, \mu)$ is the solution of system (1.9) with initial condition $x_1(0, \mu) = x_0$. Furthermore, by virtue of Condition 4, there is a first time $t_1 = t_1(\mu) > 0$ such that $x_1(t_1, \mu) \in l_1$. It follows (see [6]) that, for t asymptotically close to t_1, there is a "fall" from the manifold Γ_- and an interval of fast motion begins. In turn, this interval can be partitioned into three subintervals: a flight interval, a turning interval, and a falling interval.

On the flight segment, by virtue of the inequality

$$g(x) - h(y) > 0 \quad \forall x \in l_1, \quad \forall y > y_*,$$

which follows from Conditions 1 and 2, the variable y increases up to asymptotically large values. More precisely, phase point (1.19) moves asymptotically fast in an asymptotically small neighborhood of the interval

$$\{(x, y) : x = x_1(\mu), y_* \le y \le \varepsilon^{-3/8}\}, \tag{1.20}$$

where $x_1(\mu) = x_1(t_1(\mu), \mu) \in l_1$. A characteristic feature of this motion is that any portion of interval (1.20) corresponding to the interval $\bar{y} \le y \le \bar{\bar{y}}$, where $y_* < \bar{y} < \bar{\bar{y}}$, is traveled over a time on the order of ε. The value $y = \varepsilon^{-3/8}$ is reached by the component $y(t, \varepsilon, \mu)$ in a time of order $\varepsilon^{5/8}$.

The turning interval corresponds to the values of t at which phase point (1.19) is in the half-space $\{(x, y) : y \ge \varepsilon^{-3/8}\}$. To obtain a qualitative description of this interval, we make the substitutions $y = u\varepsilon^{-1/2}$ and $t - t_1(\mu) = \theta\varepsilon^{1/2}$ in system (1.2) and use expansion (1.4). Finally, dropping the asymptotically small terms in ε yields the Cauchy problem

$$\frac{dx}{d\theta} = u^2 f_2(x), \quad \frac{du}{d\theta} = g(x) - z_{**}, \quad x|_{\theta = 0} = x_1(\mu), \quad u|_{\theta = 0} = 0. \tag{1.21}$$

Setting $d\tau = u^2 d\theta$ in (1.21), we see that, on the turning interval, the phase point $(x(\tau, \varepsilon, \mu), u(\tau, \varepsilon, \mu))$ obtained from (1.19) after making the above substitutions moves in an asymptotically small neighborhood of the curve

$$\{(x, u) : x = x(\tau, \mu), u = u(\tau, \mu), \tau \ge 0\} \cap \{(x, u) : u \ge 0\}, \tag{1.22}$$

where

$$x(\tau, \mu) = x(t, x_0)|_{t = \tau, x_0 = x_1(\mu)}, \quad u(\tau, \mu) = (3a(t, x_0))^{1/3}|_{t = \tau, x_0 = x_1(\mu)}, \tag{1.23}$$

and $x(t, x_0)$ and $a(t, x_0)$ are functions defined above (see (1.10)–(1.13)).

Note that, in view of (1.23) and the properties of $a(t, x_0)$, the a priori condition $u \ge 0$ in (1.22) is satisfied only on the interval $0 \le \tau \le \bar{\tau}(\mu)$, where $\bar{\tau}(\mu) = t_{**}(x_0)|_{x_0 = x_1(\mu)}$ and $t_{**}(x_0)$ is the time from (1.13). More precisely, the component $u(\tau, \mu)$ has the properties

$$u(\tau, \mu) > 0 \quad \text{for} \quad 0 < \tau < \bar{\tau}(\mu), \quad u(0, \mu) = u(\bar{\tau}(\mu), \mu) = 0. \tag{1.24}$$

This behavior of $u(\tau, \mu)$ means that, on the turning interval, the component $y(t, \varepsilon, \mu)$ increases from $\varepsilon^{-3/8}$ to values on the order of $\varepsilon^{-1/2}$ and then falls to the precious level $y = \varepsilon^{-3/8}$. Note that the travel time through this interval uniformly in μ is on the order of $\varepsilon^{1/2}$.

After turning, there is a falling interval, which is similar to the above flight interval. Specifically, trajectory (1.19) on this interval is asymptotically close to the interval

$$\{(x, y) : x = x_2(\mu), \Phi_-(x_2(\mu)) \le y \le \varepsilon^{-3/8}\}, \tag{1.25}$$

where $x_2(\mu) = x(\tau, \mu)|_{\tau = \bar{\tau}(\mu)}$. Note that, by the definition of the functions $x(\tau, \mu)$ and $\bar{\tau}(\mu)$, it follows from (1.23) and (1.24) that $x_2(\mu) = \Pi_0(x_1(\mu))$, where Π_0 is operator (1.13). Thus, we automatically have $x_2(\mu) \in l_3$. Since

$$g(x) - h(y) < 0 \quad \forall x \in l_3, \quad \forall y > \Phi_-(x),$$

the component $y(t, \varepsilon, \mu)$ decreases in the motion along interval (1.25) (hence the name—the falling interval). As in the case of the flight interval, the length of this interval is on the order of $\varepsilon^{5/8}$.

After falling, there is another interval of slow motion. On this interval, phase point (1.19) is in an asymptotically small neighborhood of the curve $(x_2(t, \mu), y_2(t, \mu))$, where $y_2(t, \mu) = \Phi_-(x_2(t, \mu))$ and $x_2(t, \mu)$ is the solution of the Cauchy problem for system (1.9) with initial condition $x_2|_{t = t_1(\mu)} = x_2(\mu)$. Depending on the sign of μ, the following two scenarios are possible.

Assume first that μ is fixed and positive. Then, as was said above, system (1.9) has no cycles and, in a finite, but rather long time (on the order of $1/\sqrt{\mu}$), any of its solutions with an initial value lying on the curve l_3 reaches l_1. The last means that there exists a time $t = t_2 > t_1$ such that, at $t \simeq t_2$, trajectory (1.19) falls off Γ_-. Then another interval of fast motion follows, etc. Clearly, in the case under consideration, the described process of changing intervals of fast and slow motions continues to infinity; i.e., undamped nonclassical relaxation oscillations occur in system (1.2).

Now let μ be fixed and negative. Then it follows from the results of [7] that, for all sufficiently small $\varepsilon > 0$, the cycles $L_\pm(\mu)$ of system (1.9) are associated with a stable and an unstable cycle $\tilde{L}_-(\varepsilon, \mu)$ and $\tilde{L}_+(\varepsilon, \mu)$ in the original system (1.2); moreover,

$$\tilde{L}_\pm(0, \mu) = \{(x, y) : y = \Phi_-(x), x \in L_\pm(\mu)\}.$$

Note that, by Condition 4, for $t > t_1(\mu)$, trajectory (1.19) surely belongs to the basin of attraction of $\tilde{L}_-(\varepsilon, \mu)$. Thus, in this case, there are no self-exciting relaxation oscillations in system (1.2).

The above qualitative consideration implies that the parameter μ has a critical value $\mu_*(\varepsilon)$, $\mu_*(0) = 0$, at which system (1.2) passes from smooth self-exciting oscillations (i.e., the stable cycle $\tilde{L}_-(\varepsilon, \mu)$) to self-exciting relaxation oscillations. More precisely, the case $\mu = \mu_*(\varepsilon)$ corresponds to a saddle-node bifurcation causing the cycles $\tilde{L}_\pm(\varepsilon, \mu)$ to merge and vanish. A blue sky catastrophe in system (1.2) is observed at

$$\mu = \mu_*(\varepsilon) + \nu, \quad 0 < \nu \ll 1 \tag{1.26}$$

under an additional condition, which will be discussed later.

2. BASIC CONSTRUCTIONS

First, we show that the critical value $\mu_*(\varepsilon)$ of the parameter μ in (1.26) exists.

Lemma 1. *Given any positive integer* k, *there is a sufficiently small* $\varepsilon_k > 0$ *such that on the interval* $0 \le \varepsilon \le \varepsilon_k$ *there exists a unique function* $\mu_*(\varepsilon)$, $\mu_*(0) = 0$, *that is* C^k*-smooth in* $\sqrt{\varepsilon}$ *and has the following property*: $\forall \varepsilon \in (0, \varepsilon_0]$ *at* $\mu = \mu_*(\varepsilon)$ *system* (1.2) *has a semistable simple saddle-node cycle*

$$\tilde{L}_*(\varepsilon) : x = x_*(\varphi, \varepsilon), \quad y = y_*(\varphi, \varepsilon), \quad \frac{d\varphi}{dt} = \omega_*(\varepsilon). \tag{2.1}$$

Here, $x_*(\varphi, \varepsilon)$ *and* $y_*(\varphi, \varepsilon)$ *are* C^k*-smooth functions of* $(\varphi, \sqrt{\varepsilon})$ *and the frequency* $\omega_*(\varepsilon)$ *is a* C^k*-smooth function of* $\sqrt{\varepsilon}$; *they satisfy the equalities*

$$x_*(\varphi + 2\pi, \varepsilon) \equiv x_*(\varphi, \varepsilon), \quad y_*(\varphi + 2\pi, \varepsilon) \equiv y_*(\varphi, \varepsilon),$$
$$x_*(\varphi, 0) = x_0(\varphi), \quad y_*(\varphi, 0) = \Phi_-(x_0(\varphi)), \quad \omega_*(0) = \omega_0,$$

where the vector function $x_0(\varphi)$ *and the frequency* ω_0 *are given by* (1.14).

Proof. For an arbitrarily fixed sufficiently small $\delta_0 > 0$, in the plane (x_1, x_2), we consider a neighborhood of cycle (1.14) of the form

$$\mathcal{U} = \bigcup_{\tilde{x} \in L_0} O(\tilde{x}, \delta_0), \tag{2.2}$$

where $O(\tilde{x}, \delta_0)$ is an open ball of radius δ_0 centered at the point $\tilde{x}$. It is well known (see, e.g., [8]) that, given any positive integer k, there are sufficiently small $\varepsilon_k > 0$ and $\mu_k > 0$ such that, for all $(\varepsilon, \mu) \in [0, \varepsilon_k] \times$

$[-\mu_k, \mu_k]$, system (1.2) has an exponentially orbitally stable (with an exponent on the order of ε^{-1}) invariant slow integral manifold

$$\{(x, y) : y = H(x, \varepsilon, \mu), x \in \mathcal{U}\}, \tag{2.3}$$

where $H(x, \varepsilon, \mu)$ with $H(x, 0, \mu) = \Phi_-(x)$ is a C^k-smooth function of variables $(x, \sqrt{\varepsilon}, \mu)$. This means that the proof of the lemma is reduced to analyzing a two-dimensional system on manifold (2.3) that has the form

$$\dot{x} = f(x, H(x, \varepsilon, \mu), \mu), \quad x \in \mathcal{U}. \tag{2.4}$$

Consider a first return map

$$s \longrightarrow R(s, \varepsilon, \mu) \tag{2.5}$$

on trajectories of system (2.4). It is defined on the above-introduced secant Σ (see Fig. 4) and is a C^k-smooth extension of map (1.16) with respect to $\sqrt{\varepsilon}$. Consider the system of equations

$$R(s, \varepsilon, \mu) = s, \quad R'_s(s, \varepsilon, \mu) = 1 \tag{2.6}$$

for finding s and μ as functions of ε. It is easy to see that, at $\varepsilon = 0$, this system has the solution $(s, \mu) = (0, 0)$ and its Jacobian with respect to s and μ is nonzero at the point $(s, \varepsilon, \mu) = (0, 0, 0)$; more specifically, it is equal to $-2\alpha_0 d_0$, where α_0 and d_0 are the constants from (1.17) and (1.18). Thus, by the implicit function theorem, system (2.6) has a unique C^k-smooth (in $\sqrt{\varepsilon}$) solution $(s_*(\varepsilon), \mu_*(\varepsilon))$, $s_*(0) = \mu_*(0) = 0$.

The meaning of the above constructions is clear: at $\mu = \mu_*(\varepsilon)$, map (2.5) has a fixed point $s = s_*(\varepsilon)$ of the simple saddle-node type associated in system (2.4) with a semistable cycle

$$L_*(\varepsilon) : x = x_*(\varphi, \varepsilon), \quad x_*(\varphi + 2\pi, \varepsilon) \equiv x_*(\varphi, \varepsilon), \quad \frac{d\varphi}{dt} = \omega_*(\varepsilon), \tag{2.7}$$

which is a C^k-smooth extension of saddle-node cycle (1.14) with respect to $\sqrt{\varepsilon}$. Cycle (2.1) is obtained from (2.7) by adding the component $y = y_*(\varphi, \varepsilon)$, where $y_*(\varphi, \varepsilon) = H(x_*(\varphi, \varepsilon), \varepsilon, \mu_*(\varepsilon))$. Lemma 1 is proved.

In addition to this lemma, we note that, by Conditions 1–5, each trajectory of system (1.2) at $\mu = \mu_*(\varepsilon)$ that belongs to the unstable manifold $W^u_{\text{loc}}(\tilde{L}_*(\varepsilon))$ of cycle (2.1), again tends, as $t \longrightarrow +\infty$, to $\tilde{L}_*(\varepsilon)$, hitting the node domain of this cycle. Thus, $\mu_*(\varepsilon)$ is the desired critical value corresponding to the blue sky catastrophe.

At the following stage, we consider the system

$$\dot{x} = F_*(x, \varepsilon, \nu) \tag{2.8}$$

obtained from (2.4) taking into account (1.26). Let us transform it into a form as simple as possible. For this purpose, in domain (2.2), we introduce radial and cyclic coordinates $(s, \varphi) : |s| \le s_0$, $s_0 = \text{const} > 0$, $0 \le \varphi \le 2\pi (\text{mod}\, 2\pi)$ tied to cycle (2.7). More precisely, φ is defined as the angular coordinate from (2.7), while s denotes a parameter on the segment $\Sigma(\varphi, \varepsilon)$ of the normal to the curve $L_*(\varepsilon)$ through the point $x_*(\varphi, \varepsilon)$ (this parameter is defined in Section 1). The results of [9] imply the existence of a sufficiently small $\rho_0 > 0$ and C^k-smooth functions $h_j(\rho, \theta, \varepsilon)$ and $h_j(\rho, \theta + 2\pi, \varepsilon) \equiv h_j(\rho, \theta, \varepsilon)$ $(j = 1, 2)$ of variables $(\rho, \theta, \sqrt{\varepsilon}) \in [-\rho_0, \rho_0] \times [0, 2\pi] \times [0, \sqrt{\varepsilon_k}]$ such that, after passing to the coordinates (s, φ) and making the substitutions $s = \rho + \rho^2 h_1(\rho, \theta, \varepsilon)$ and $\varphi = \theta + \rho h_2(\rho, \theta, \varepsilon)$, system (2.8) with $\nu = 0$ becomes

$$\dot{\rho} = d_1(\varepsilon)\rho^2 + d_2(\varepsilon)\rho^3, \quad \dot{\theta} = \omega_*(\varepsilon), \tag{2.9}$$

where $\omega_*(\varepsilon)$ is the frequency from (2.7) and $d_j(\varepsilon)$, $j = 1, 2$, are C^k-smooth functions of $\sqrt{\varepsilon}$; moreover, $d_1(0) = d_0/2\pi > 0$, where d_0 is the constant from (1.17).

Now assume that $\nu > 0$ in system (2.8). Then the above substitutions transform it to a form similar to (2.9), namely,

$$\dot{\rho} = d_1(\varepsilon)\rho^2 + d_2(\varepsilon)\rho^3 + \nu\Delta_1(\rho, \theta, \varepsilon, \nu), \quad \dot{\theta} = \omega_*(\varepsilon) + \nu\Delta_2(\rho, \theta, \varepsilon, \nu), \tag{2.10}$$

where $\Delta_j(\rho, \theta, \varepsilon, \nu)$, $j = 1, 2$ are functions that are C^k-smooth in $(\rho, \theta, \sqrt{\varepsilon}, \nu)$ and 2π-periodic in θ. A further simplification of system (2.10) is related to polynomial (in ρ and ν) changes of variables, due to which the dependence of Δ_j on θ can be weakened, more exactly, the functions $\partial\Delta_j/\partial\theta$, $j = 1, 2$, can be made to vanish at $\rho = 0$, $\nu = 0$ together with any prescribed number of their derivatives with respect to ρ and ν. Specifically, there exists a change of variables

$$\rho = r + \nu h_{1,1}(\psi, \varepsilon) + \nu r h_{1,2}(\psi, \varepsilon), \quad \theta = \psi + \nu h_{2,1}(\psi, \varepsilon) + \nu r h_{2,2}(\psi, \varepsilon),$$

where $h_{j,s}(\psi, \varepsilon)$, $h_{j,s}(\psi + 2\pi, \varepsilon) \equiv h_{j,s}(\psi, \varepsilon)$ $(j, s = 1, 2)$ are C^k-smooth functions of ψ and $\sqrt{\varepsilon}$ (a method for constructing them is described, e.g., in [10]), that reduces system (2.10) to the form

$$\begin{aligned} \dot{r} &= d_1(\varepsilon)r^2 + d_2(\varepsilon)r^3 + \alpha_1(\varepsilon)\nu + \alpha_2(\varepsilon)\nu r + \nu\Delta_1(r, \psi, \varepsilon, \nu), \\ \dot{\psi} &= \omega_*(\varepsilon) + \beta_1(\varepsilon)\nu + \beta_2(\varepsilon)\nu r + \nu\Delta_2(r, \psi, \varepsilon, \nu). \end{aligned} \tag{2.11}$$

Here, $\alpha_j(\varepsilon)$ and $\beta_j(\varepsilon)$, $j = 1, 2$, are scalar functions that are C^k-smooth in $\sqrt{\varepsilon}$ and the remainders Δ_j $(j = 1, 2)$ in (2.11) satisfy the same general periodicity and smoothness conditions as their counterparts from (2.10). Now, however, they satisfy the equalities

$$\Delta_j(0, \psi, \varepsilon, 0) \equiv \frac{\partial\Delta_j}{\partial r}(0, \psi, \varepsilon, 0) \equiv 0, \quad j = 1, 2.$$

Note also the property $\alpha_1(0) = \alpha_0/2\pi > 0$, where α_0 is constant (1.18), which will be used in what follows.

The subsequent analysis is based on some additional geometric constructions. Specifically, we consider two two-dimensional cylindrical surfaces $S_\pm$ defined in the variables (r, ψ, v), $v = y - H(x, \varepsilon, \mu)$ by the equalities

$$S_\pm = \{(r, \psi, v) : r = \pm r_0, 0 \le \psi \le 2\pi(\mathrm{mod}\, 2\pi), |v| \le v_0\}, \tag{2.12}$$

where r_0, $v_0 > 0$ are fixed sufficiently small constants. Note that, under condition (1.26), surfaces (2.12) have no contacts with the trajectories of system (1.2), since the derivatives $\dot{r}|_{r = \pm r_0}$ are positive by virtue of this system (this fact follows from the positiveness of the coefficients $d_1(\varepsilon)$ and $\alpha_1(\varepsilon)$ of normal form (2.11)). Moreover, the qualitative behavior of solutions (1.19) described in Section 1 implies that, first, a correspondence operator $\Pi_{+,-}(\varepsilon, \nu) : S_+ \longrightarrow S_-$ on trajectories of system (1.2) is well defined, and, second, this operator has a finite limit as $\varepsilon \longrightarrow 0$, $\nu \longrightarrow 0$:

$$\Pi_{+,-}(0, 0) : (\psi, v) \longrightarrow (\gamma(\psi), 0), \tag{2.13}$$

where $\gamma(\psi) \in C^\infty$ is a 2π-periodic function.

Let us describe in more detail how to determine the function $\gamma(\psi)$ in (2.13). In the (x_1, x_2) plane, consider two closed curves $L_{(\pm)}$ given in the variables (r, ψ) by

$$L_{(\pm)} = \{(r, \psi) : r = \pm r_0, 0 \le \psi \le 2\pi(\mathrm{mod}\, 2\pi)\}$$

(see Fig. 5, where the dashed line depicts the cycle L_0 laying between $L_{(-)}$ and $L_{(+)}$). Fix an arbitrary point on $L_{(+)}$ corresponding to an angle ψ and consider the solution of system (1.9) with $\mu = 0$ starting at this point. By virtue of condition (1.15), there is a one-to-one correspondence between $L_{(+)}$ and l_1 on trajectories of this system. Accordingly, assume that the curve l_1 is parametrized by the same coordinate ψ as $L_{(+)}$. More precisely, assume that points from $L_{(+)}$ and l_1 lying on the same trajectory are associated with identical values of ψ (see Fig. 5).

Now, given a point on l_1 with a coordinate ψ (denoted by x_0), consider a segment of the trajectory of system (1.7) starting at x_0 and corresponding to $0 \le t \le t_{**}(x_0)$, where $t_{**}(x_0)$ is the time from (1.13). This segment arrives at the point $x_1 = \Pi_0(x_0)$ on the curve l_3. Since l_3 is the image of l_1 under diffeomorphism (1.13), we can again assume that the points x_0 and x_1 have the same coordinate ψ (see Fig. 5).

To complete the construction of $\gamma(\psi)$, we consider the trajectory of system (1.9) at $\mu = 0$ with an initial condition at the point $x_1 \in l_3$ obtained at the preceding step (see Fig. 5). Once again invoking Condition 4, we see that, as t increases, this trajectory intersects (in the generic case) the curve $L_{(-)}$ at a point with coordinate $\psi_1 = \gamma(\psi)$, where $\gamma(\psi) \in C^\infty$. Since l_3 does not enclose the singular point $x = \bar{x}$ (see Fig. 5), $\gamma(\psi)$ has the periodicity property of the first kind; i.e., $\gamma(\psi + 2\pi) \equiv \gamma(\psi)$.

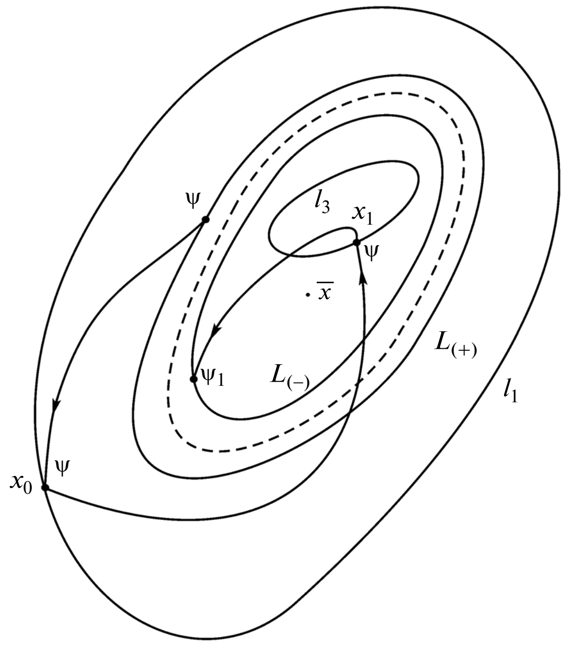

Fig. 5.

The following assertion is derived if the general results of [6] concerning classical and nonclassical relaxation oscillations are applied to system (1.2).

Lemma 2. *If Conditions* 1–5 *and relation* (1.26) *hold, then*

$$\lim_{\varepsilon \to 0, \nu \to 0} \Pi_{+,-}(\varepsilon, \nu) = \Pi_{+,-}(0, 0) \tag{2.14}$$

in the metric of C^k *with respect to* $(\psi, v) \in [0, 2\pi] \times [-v_0, v_0]$ *for any fixed k*.

To complete the examination of the behavior of solutions to system (1.2) with condition (1.26), we consider the correspondence operator $\Pi_{-,+}(\varepsilon, \nu)$ on system's trajectories acting from S_- to S_+. First, we show that this operator exists for any $0 < \nu \ll 1$. For this purpose, making the substitution $y - H(x, \varepsilon, \mu) = v$ in (1.2) and taking into account (1.26), we derive a system of the form

$$\dot{x} = F(x, v, \varepsilon, \nu), \quad \varepsilon \dot{v} = G(x, v, \varepsilon, \nu) v, \quad (x, v) \in \mathcal{U} \times [-v_0, v_0]. \tag{2.15}$$

Here, G and F are C^k-smooth functions of the variables $(x, v, \sqrt{\varepsilon}, \nu)$ such that

$$F(x, 0, \varepsilon, \nu) = F_*(x, \varepsilon, \nu), \quad G(x, 0, 0, 0) = -h'(\Phi_-(x)) < 0, \tag{2.16}$$

where F_* is the right-hand side of system (2.8).

Property (2.16) of G implies that, in any finite time $t_0 = \text{const} > 0$, the component v of any solution to system (2.15) with an initial value at $t = 0$ lying on the surface S_-, becomes a quantity of order $\exp(-c_0/\varepsilon)$, $c_0 = \text{const} > 0$, in absolute value. Thus, the existence of the operator $\Pi_{-,+}(\varepsilon, \nu)$ is reduced to the existence of the correspondence operator $\Pi^*_{-,+}(\varepsilon, \nu) : L_{(-)} \longrightarrow L_{(+)}$ on trajectories of system (2.11), which is obtained from (2.15) at $v = 0$ by passing to the coordinates r and ψ. The latter operator is well defined, since the right-hand side of the equation for r in (2.11) is strictly positive for $\nu > 0$ and $r \in [-r_0, r_0]$. Moreover, the following result holds for $\Pi^*_{-,+}(\varepsilon, \nu)$.

Lemma 3. *Given any positive integer k, there are sufficiently small constants* $\varepsilon_k > 0$ *and* $\nu_k > 0$ *such that, for all* $0 < \nu \le \nu_k$ *and* $0 \le \varepsilon \le \varepsilon_k$, *the operator* $\Pi^*_{-,+}(\varepsilon, \nu)$ *can be represented as*

$$\Pi^*_{-,+}(\varepsilon, \nu) : \psi \longrightarrow \psi + c_*(\varepsilon, \nu)/\sqrt{\nu} + \sqrt{\nu}\Psi(\psi, \varepsilon, \nu) (\text{mod}\, 2\pi), \tag{2.17}$$

where $c_*(\varepsilon, \nu)$ *with* $c_*(0, 0) > 0$ *is a continuous function of its variables* (*down to the value* $\nu = 0$) *and* ψ *is a* 2π*-periodic function of* $\Psi(\psi, \varepsilon, \nu)$ *that is continuous and bounded on the set* $(\psi, \varepsilon, \nu) \in [0, 2\pi] \times [0, \varepsilon_k] \times (0, \nu_k)$ *together with its derivatives* $\partial^m\Psi/\partial\psi^m$, $m \le k$.

The proof of this lemma is omitted, since it is the same as the proof of Lemma 3 in [4].

Returning to the original operator $\Pi_{-,+}(\varepsilon, \nu)$, we conclude that the above constructions imply the following result.

Lemma 4. *For any positive integer k, there are sufficiently small* $\varepsilon_k > 0$ *and* $\nu_k > 0$ *such that, for all* $0 < \nu \le \nu_k$ *and* $0 \le \varepsilon \le \varepsilon_k$, *the operator* $\Pi_{-,+}(\varepsilon, \nu)$ *can be represented as*

$$\Pi_{-,+}(\varepsilon, \nu): \begin{cases} \psi \longrightarrow \psi + c_*(\varepsilon, \nu)/\sqrt{\nu} + \Psi_1(\psi, v, \varepsilon, \nu)(\mathrm{mod}\, 2\pi), \\ v \longrightarrow \Psi_2(\psi, v, \varepsilon, \nu)\exp\left(-\dfrac{c_{**}}{\varepsilon\sqrt{\nu}}\right). \end{cases} \tag{2.18}$$

Here, the function $c_*(\varepsilon, \nu)$ *has the same properties as its counterpart in* (2.17), $c_{**} = \text{const} > 0$, *and* 2π*-periodic on* ψ function $\Psi_j, j = 1, 2$, *and their all possible partial derivatives on* ψ, v *to the order to are continuous on* $(\psi, v, \varepsilon, \nu) \in [0, 2\pi] \times [-v_0, v_0] \times [0, \varepsilon_k] \times [0, \nu_k]$. *Moreover,* $\Psi_1(\psi, v, 0, 0) \equiv 0$.

In fact, the proof of Lemma 4 is reduced to applying Lemma 3. Indeed, in system (2.15), making the smooth change of variables $x + \varepsilon w(x, v, \varepsilon, \nu) \longrightarrow x$, which straightens the strongly stable invariant foliation over manifold (2.3) (see [10]), we obtain the triangular system

$$\dot{x} = \tilde{F}(x, \varepsilon, \nu), \quad \varepsilon\dot{v} = \tilde{G}(x, v, \varepsilon, \nu)v, \tag{2.19}$$

where $\tilde{G}$ and $\tilde{F}$ are C^k-smooth functions of $(x, \sqrt{\varepsilon}, \nu)$ and $(x, v, \sqrt{\varepsilon}, \nu)$ that differ from F_* and G involved in (2.15) and (2.16) by quantities of order ε. Then we normalize the first equation in (2.19), i.e., pass from x to the coordinates $(\tilde{r}, \tilde{\psi})$, in which this equation becomes similar to (2.11). Next, we consider secants $\tilde{S}_\pm$ defined by analogy with (2.12) and define a correspondence operator $\tilde{\Pi}_{-,+}(\varepsilon, \nu): \tilde{S}_- \longrightarrow \tilde{S}_+$ on trajectories of system (2.19).

The structure of Eqs. (2.19) and Lemma 3 imply that the operator $\tilde{\Pi}_{-,+}(\varepsilon, \nu)$ also has a triangular form; more exactly, it can be represented as

$$\tilde{\Pi}_{-,+}(\varepsilon, \nu): \begin{cases} \tilde{\psi} \longrightarrow \tilde{\psi} + c_*(\varepsilon, \nu)/\sqrt{\nu} + \sqrt{\nu}\tilde{\Psi}_1(\tilde{\psi}, \varepsilon, \nu), \\ v \longrightarrow \tilde{\Psi}_2(\tilde{\psi}, v, \varepsilon, \nu)\exp\left(-\dfrac{c_{**}}{\varepsilon\sqrt{\nu}}\right). \end{cases} \tag{2.20}$$

Here, $c_{**} = \text{const} > 0$, the functions $c_*(\varepsilon, \nu)$ and $\tilde{\Psi}_1(\tilde{\psi}, \varepsilon, \nu)$ have the properties described in Lemma 3, and $\tilde{\Psi}_2$ is a 2π-periodic function of $\tilde{\psi}$ that is continuous in all its variables

$$(\tilde{\psi}, v, \varepsilon, \nu) \in [0, 2\pi] \times [-v_0, v_0] \times [0, \varepsilon_k] \times [0, \nu_k]$$

together with its derivatives with respect to $(\tilde{\psi}, v)$ up to the order k. Note that the exponential smallness of v as indicated in (2.20) follows from the form of the second equation in system (2.19), from the property $\tilde{G}(x, 0, 0, 0) = G(x, 0, 0, 0) < 0$ (see (2.16)), and from the fact that any trajectory of this system stays between the secants $\tilde{S}_\pm$ for the time

$$T \sim \int_{-r_0}^{r_0} d\tilde{r}/\delta(\tilde{r}, \tilde{\psi}, \varepsilon, \nu) = O(1/\sqrt{\nu}),$$

where $\delta(\tilde{r}, \tilde{\psi}, \varepsilon, \nu) = O(\tilde{r}^2 + \nu)$ is the right-hand side of the equation for $\tilde{r}$, which is similar to the right-hand side of the equation for r in (2.11).

To complete the proof of Lemma 4, we note that, after we pass from $(\tilde{r}, \tilde{\psi})$ to the original coordinates (r, ψ), map (2.20) takes the required form (2.18).

3. RESULTS

Relying on the constructions made in Section 2, we can define the Poincaré first return operator

$$\Pi(\varepsilon, \nu) = \Pi_{-,+}(\varepsilon, \nu) \circ \Pi_{+,-}(\varepsilon, \nu) : S_+ \longrightarrow S_+ \tag{3.1}$$

on trajectories of system (1.2) with condition (1.26). Relations (2.14) and (2.18) imply the following assertion, which is the main result of this paper.

Theorem 1. *For any positive integer k, there are sufficiently small* $\varepsilon_k > 0$ *and* $\nu_k > 0$ *such that, for* $0 \le \varepsilon \le \varepsilon_k$ *and* $0 < \nu \le \nu_k$, *operator* (3.1) *has the form*

$$\Pi(\varepsilon, \nu) : \begin{cases} \psi \longrightarrow c_*(\varepsilon, \nu)/\sqrt{\nu} + \gamma(\psi) + \Lambda_1(\psi, v, \varepsilon, \nu) (\mathrm{mod}\, 2\pi), \\ v \longrightarrow \Lambda_2(\psi, v, \varepsilon, \nu) \exp\left(-\dfrac{c_{**}}{\varepsilon\sqrt{\nu}}\right). \end{cases} \tag{3.2}$$

Here, $\gamma(\psi)$ *and* $c_*(\varepsilon, \nu)$ *are the functions from* (2.13) *and* (2.18), *respectively*; $c_{**} = \mathrm{const} > 0$; *and* Λ_j, $j = 1, 2$, *are* 2π-*periodic functions of* ψ *that are continuous in* $(\psi, v, \varepsilon, \nu) \in [0, 2\pi] \times [-v_0, v_0] \times [0, \varepsilon_k] \times [0, \nu_k)$ *together with their derivatives with respect to* (ψ, v) *up to the order k inclusive. Moreover*, $\Lambda_1(\psi, v, 0, 0) \equiv 0$.

Relying on representation (3.2), it is easy to see that $\Pi(\varepsilon, \nu)S_+ \subset S_+$, which means that map (3.1) has the maximum attractor

$$A_{\max} = \bigcap_{n=0}^{\infty} \Pi^n(\varepsilon, \nu) S_+. \tag{3.3}$$

It is also clear that, in the first approximation, the structure of set (3.3) is determined by the one-dimensional map

$$\Pi_\kappa : \psi \longrightarrow \tilde{\psi} = \kappa + \gamma(\psi), \tag{3.4}$$

where $\kappa = c_*(\varepsilon, \nu)/\sqrt{\nu}$; this map is obtained from (3.2) by dropping the terms that are asymptotically small in ε and ν. In what follows, this map is considered assuming that κ is an independent parameter ranging over the entire number line $\mathbb{R}$.

When analyzing attractor (3.3), we consider only two major cases. Let us begin with the simpler one, namely, assume that

$$|\gamma'(\psi)| < 1 \quad \forall \psi \in [0, 2\pi]. \tag{3.5}$$

Then it is easy to see that map (3.4) has a unique fixed point

$$\psi = \psi_0(\kappa), \quad \psi_0(\kappa + 2\pi) \equiv \psi_0(\kappa) + 2\pi, \tag{3.6}$$

which depends continuously on κ. Obviously, under condition (3.5), the original map (3.2) is a contraction. Thus, the following result holds in the case under consideration.

Theorem 2 (on blue sky catastrophe). *Let inequality* (3.5) *hold. Then attractor* (3.3) *consists of a unique exponentially stable fixed point* $(\psi, v) = (\psi(\varepsilon, \nu), v(\varepsilon, \nu))$ *whose components have the following* ε-*uniform asymptotic representations as* $\nu \longrightarrow 0$:

$$\psi(\varepsilon, \nu) = \psi_0(\kappa)\big|_{\kappa = c_*(\varepsilon, \nu)/\sqrt{\nu}} + o(1), \quad v(\varepsilon, \nu) = O\left(\exp\left(-\frac{c_{**}}{\varepsilon\sqrt{\nu}}\right)\right), \tag{3.7}$$

where $c_{**} > 0$ *is the constant from* (3.2) *and* $\psi_0 = \psi_0(\kappa)$ *is function* (3.6).

It should be noted that Theorem 2 guarantees the feasibility of a blue sky catastrophe of interest in system (1.2) with condition (1.26). Indeed, the fixed point (3.7) of operator (3.1) is associated with a stable relaxation cycle $\tilde{L}(\varepsilon, \nu)$ of this system such that its period and length ε-uniformly tend to infinity as $\nu \longrightarrow 0$. Moreover, by virtue of the first equality in (3.7), the set of all partial limits of the component $\psi(\varepsilon, \nu)$ modulo 2π coincides, as $\nu \longrightarrow 0$, with the interval $[0, 2\pi]$. This means that the upper topological limit of the

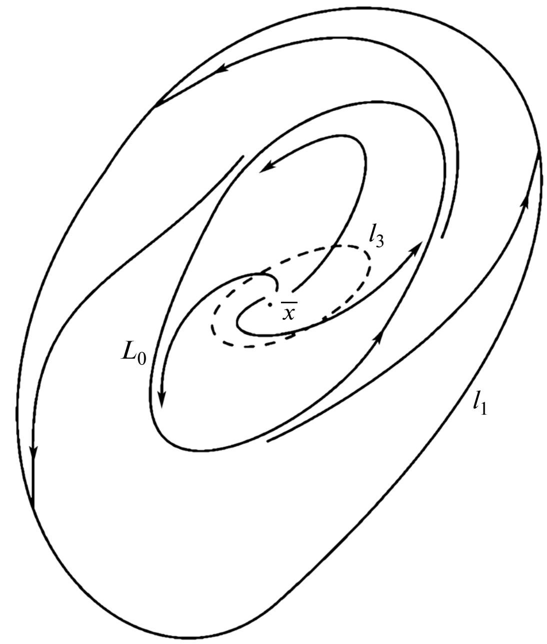

Fig. 6.

curve $\tilde{L}(\varepsilon, \nu)$ as $\nu \longrightarrow 0$ is equal to $W^u(\tilde{L}_*(\varepsilon)) \cup \tilde{L}_*(\varepsilon)$, where $\tilde{L}_*(\varepsilon)$ is the saddle-node cycle (2.1) and W^u is its unstable manifold extended along trajectories of system (1.2) at $\mu = \mu_*(\varepsilon)$, which consists of solutions doubly asymptotic to $\tilde{L}_*(\varepsilon)$, as was mentioned in Section 2.

Now assume that the curve l_3 in Condition 4 lies as before in the domain bounded by the cycle L_0 but encloses the equilibrium $x = \bar{x}$ and has no contacts with the trajectories of system (1.9) at $\mu = 0$ (see Fig. 6). Then the function $\gamma(\psi)$ in (2.13) has the properties

$$\gamma(\psi + 2\pi) \equiv \gamma(\psi) + 2\pi, \quad \gamma'(\psi) > 0 \quad \forall \psi \in [0, 2\pi]. \tag{3.8}$$

In this case, all the above constructions and Theorem 1 remain valid. Moreover, the following result is true.

Theorem 3. *Let Condition* 4 *be modified as described above. Then attractor* (3.3) *of map* (3.2) *consists of the invariant curve*

$$\left\{(\psi, v) : v = V(\psi, \varepsilon, \nu)\exp\left(-\frac{c_{**}}{\varepsilon\sqrt{\nu}}\right)\right\}. \tag{3.9}$$

Here, $c_{**} > 0$ *and* $V(\psi, \varepsilon, \nu)$ *is a* 2π*-periodic function of* ψ *that has the same continuity and smoothness properties as the function* $\Lambda_2(\psi, 0, \varepsilon, \nu)$ *in* (3.2).

As proof, we note that, in view of properties (3.8), the ring principle (see [10, 11]) can be applied to map (3.2). This principle implies that a unique globally exponentially stable invariant curve of form (3.9) exists in the set S_+. Note also that this curve is associated with a stable two-dimensional invariant torus in the original system (1.2) with condition (1.26).

4. APPLICATION TO MODELING OF CARDIAC RHYTHMS

It is well known that the cardiac signal is a nonstationary and nonharmonic signal. It depends on the individual biological properties of a particular person and consists of waves, intervals, and segments. The waves are local deflections (extrema) in ECG. The segment is the portion of a cardiac signal between two consecutive waves, and the interval consists of a wave (complex of waves) and a segment. A sample ECG is presented in Fig. 7 (taken from [12]). According to this graph, the following basic components of ECG can be distinguished (see [12, 13]).

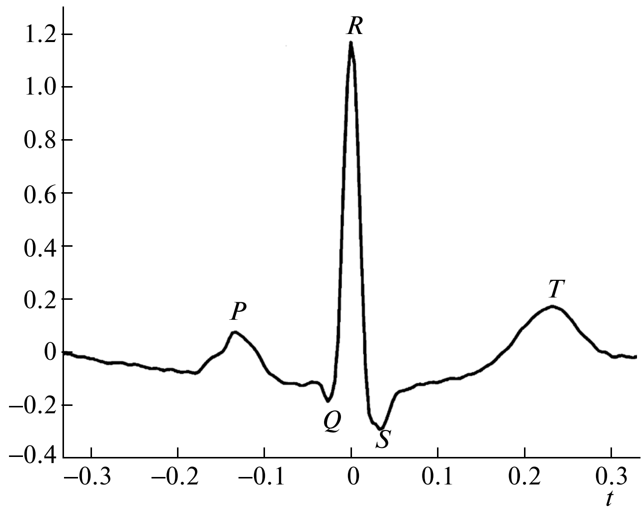

Fig. 7.

(i) The P wave represents the contraction of the atria (atrial depolarization). It begins in pacemaker cells of the sinus node and spreads to the right and left atria.

(ii) The PQ interval reflects signal propagation trough the atriums and atrioventricular junction to the myocardium of the ventricles.

(iii) The QRS complex is caused by the contraction of the ventricles (ventricular depolarization). It consists of three consecutive waves Q, R, and S and represents the sum of the potentials of the depolarizing muscle cells in the inner and outer layers of the myocardium.

(iv) The QT interval represents the sum of ventricular depolarization and repolarization. This interval is also known as the *electrical systole*.

(v) The ST segment corresponds to the period of time when the myocardium of the ventricles is completely involved in excitation. It precedes the last phase of the cardiac cycle, during which the cardiac muscle recovers after contraction.

(vi) The T wave represents the period of relaxation of the ventricular myocardium. In this period, the cardiac muscle is in rest.

(vii) The RR interval represents a complete cardiac cycle (i.e., the period of time between neighboring R waves).

Used by many authors, the principle underlying the modeling of cardiac rhythms consists in choosing a particular system of ordinary differential equations in which the graph of the time dependence of an ECG component is similar in shape to that shown in Fig. 7. Based on this approach, several three-dimensional models have been proposed. The most popular of them was suggested in [12]. Also worth noting are the Zeeman model [14] and the three-dimensional model in [15].

There are also multidimensional models of cardiac activity, which represent systems of coupled FitzHugh–Nagumo, Van der Pol, and other equations (see, e.g., [16–18]).

Our approach to ECG modeling combines two rather nontrivial ideas: nonclassical relaxation oscillations and blue sky catastrophes. As a result, we obtain a whole class of systems of form (1.2) that can be used to describe cardiac activity. More precisely, in our case, variations in the fast variable y are responsible for the electrical activity of the cardiac muscle.

As a particular example of our model, we consider system (1.2) with

$$f_1(x, \mu) = (f_{1,1}, f_{1,2}), \quad f_2(x) = (f_{2,1}, f_{2,2}),$$

$$f_{1,1} = (x_1 - b_1)\frac{\rho}{\sqrt{1+\rho^2}} - x_2, \quad f_{1,2} = x_2\frac{\rho}{\sqrt{1+\rho^2}} + x_1 - b_1, \tag{4.1}$$

$$\rho = ((x_1 - b_2)^2 + x_2^2 - b_2^2)^2 + \mu, \quad f_{2,1} = -\alpha x_1, \quad f_{2,2} = -\alpha x_2,$$

$$g(x) = \delta_1 x_2^2 + \delta_2 x_2^2, \quad h(y) = c_1 y \exp(-y) + c_2(1 - \exp(-y))$$

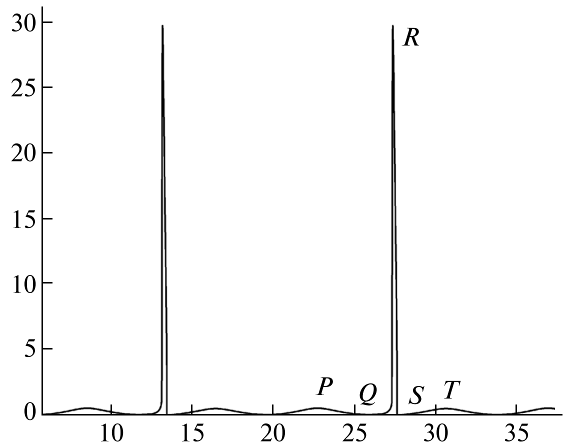

Fig. 8.

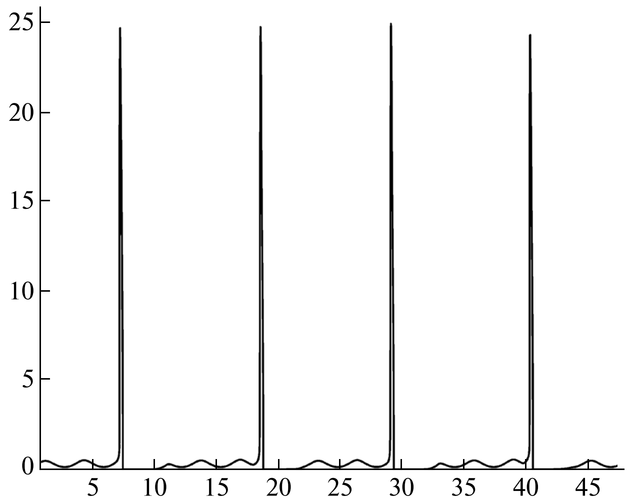

Fig. 9.

and with parameters specified as

$$b_1 = 1, \quad b_2 = 2, \quad c_1 = 28, \quad c_2 = 1, \quad \alpha = 0.5, \quad \delta_1 = \delta_2 = 1, \quad \varepsilon = 0.01, \quad \mu = 0.0044.$$

A numerical analysis has shown that this system then has a stable relaxation cycle as an attractor. Figure 8 presents the y component of this cycle against t. Comparing it with an actual ECG (see Fig. 7), we see that our plot also involves P, Q, R, S, and T waves.

It should be noted that, generally speaking, a healthy human heart is not a periodic oscillator. Most likely, the cardiac signal is quasi-periodic [13] or even chaotic [19]. A quasi-periodic variant can easily be modeled as based on the theory constructed in this paper. Indeed, consider system (1.2), (4.1) with

$$b_1 = 0, \quad b_2 = 2, \quad c_1 = 18, \quad c_2 = 1, \quad \alpha = 0.5, \quad \delta_1 = 0.6, \quad \delta_2 = 1.5, \quad \varepsilon = 0.01, \quad \mu = 0.0085.$$

In this case, we apply Theorem 3 and, hence, the system has a stable two-dimensional invariant torus. The y component corresponding to oscillations on this torus is shown in Fig. 9.

To model a chaotic cardiac signal, it is reasonable to consider the multidimensional version of system (1.2), in which $x \in \mathbb{R}^m$, $m \geq 3$, and Conditions 1–5 are properly modified. More precisely, Conditions 1 and 2 are preserved nearly completely. However, we now assume that, for any $z > 0$, the equation $g(x) = z$ determines a closed $(m - 1)$-dimensional surface of class C^∞ homeomorphic to a sphere.

Conditions 3 and 4, in contrast to the preceding two, require more substantial modifications. Specifically, we assume that system (1.9) with $\mu = 0$ has, as before, a cycle L_0 of simple saddle-node type in the domain $\Omega_1 = \{x \in \mathbb{R}^m : g(x) < z_*\}$. Furthermore, we assume that, over time, the trajectories of this system with initial values lying on the unstable manifold $W^u(L_0)$ of the given cycle intersect the surface $l_1 = \{x \in \mathbb{R}^m : g(x) = z_*\}$ but do no contact with it. Thus, the set $W^u(L_0) \cap l_1$ is a simple closed curve C_1 of class C^∞.

The constraints imposed on system (1.7) can be somewhat relaxed not only in the multidimensional case, but even at $m = 2$. Sacrificing a visual geometric representation and dropping inequality (1.8), we assume that, for any $x_0 \in C_1$, function (1.10) is defined for all $t \geq 0$ and has a root $t = t_{**}(x_0) > 0$ such that

$$a(t, x_0) > 0 \quad \text{for} \quad 0 < t < t_{**}, \quad a(t, x_0) < 0 \quad \text{for} \quad t > t_{**}, \quad \frac{\partial a}{\partial t}(t, x_0)\bigg|_{t = t_{**}} < 0. \tag{4.2}$$

Due to conditions (4.2), operator (1.13) can be well defined in a sufficiently small neighborhood of the curve C_1.

The following constraint concerns the behavior of trajectories of system (1.9) at $\mu = 0$ with initial values lying on the curve $C_2 = \Pi_0(C_1)$, where Π_0 is diffeomorphism (1.13). Assume that all these trajectories tend as $t \longrightarrow +\infty$ to a cycle L_0 and touch its two-dimensional exponentially stable center manifold $W^c(L_0)$.

To formulate an analogue of Condition 5, we consider an exponentially stable two-dimensional invariant manifold W_μ of system (1.9) that is an extension of the above manifold $W^c(L_0)$ with respect to μ. Furthermore, for the two-dimensional system that is the restriction of (1.9) to W_μ, the Poincaré first return map is defined in a similar manner to (1.16). As before, we assume that the coefficient d_0 in the Taylor expansion (1.17) of this map is strictly positive and an inequality of form (1.18) is satisfied.

The constructions made in Sections 2 and 3, which remain valid in the multidimensional case, imply that, for $x \in \mathbb{R}^m$, $m \geq 3$, the attractors of system (1.2), (1.26) (3.2) are determined by a similar map

$$\Pi(\varepsilon, \nu) : \begin{cases} \psi \longrightarrow c_*(\varepsilon, \nu)/\sqrt{\nu} + \gamma(\psi) + \Lambda_1(\psi, v_1, v_2, \varepsilon, \nu)(\operatorname{mod} 2\pi), \\ v_1 \longrightarrow \Lambda_2(\psi, v_1, v_2, \varepsilon, \nu)\exp\left(-\dfrac{c_{**}}{\varepsilon\sqrt{\nu}}\right), \\ v_2 \longrightarrow \Lambda_3(\psi, v_1, v_2, \varepsilon, \nu)\exp\left(-\dfrac{c_{***}}{\sqrt{\nu}}\right) \end{cases} \tag{4.3}$$

defined in some annulus

$$K = \{(\psi, v_1, v_2) : 0 \leq \psi \leq 2\pi(\operatorname{mod} 2\pi), |v_1| \leq v_{0,1}, \|v_2\| \leq v_{0,2}\}. \tag{4.4}$$

Here, $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}^{m-2}$, $\|*\|$ is the Euclidean norm in $\mathbb{R}^{m-2}$; $v_{0,1}$, $v_{0,2} > 0$ are sufficiently small constants; $c_*(\varepsilon, \nu)$ is a function having the same properties as its counterpart in (3.2); and $c_{**}, c_{***} = \text{const} > 0$. The functions Λ_j, $j = 1, 2, 3$, are continuous in all their derivatives (up to $\varepsilon = 0$ and $\nu = 0$) together with their partial derivatives with respect to (ψ, v_1, v_2) up to the order k inclusive (k is any prescribed positive integer). Finally, $\Lambda_1(\psi, v_1, 0, 0, 0) \equiv 0$.

As in the case $m = 2$, the attractors of map (4.3) in annulus (4.4) are eventually determined by a one-dimensional map of the circle to itself of form (3.4). In turn, the function $\gamma(\psi) \in C^\infty$ involved can be represented as

$$\gamma(\psi) = n_0\psi + \gamma_0(\psi), \quad \gamma_0(\psi + 2\pi) \equiv \gamma_0(\psi), \tag{4.5}$$

where $n_0 \in \mathbb{Z}$ is a topological invariant.

For $m = 2$, only the cases $n_0 = 0$ and $n_0 = 1$ are possible in formula (4.5). The curve l_3 does not enclose the singular point $x = \bar{x}$ in the former case (see Fig. 4) and encloses it in the latter (see Fig. 6). For each of the cases, its own result was obtained above (see Theorems 2 and 3). Of course, the results remain valid for $m \geq 3$. However, even at $m = 3$, a fundamentally new situation is possible, namely,

$$|n_0| \geq 2, \quad |n_0 + \gamma_0'(\psi)| > 1 \quad \forall \psi \in [0, 2\pi]. \tag{4.6}$$

The constructions in [1] imply that, under conditions (4.6), map (4.3) in annulus (4.4) has a chaotic hyperbolic attractor of the Smale–Williams solenoid type.

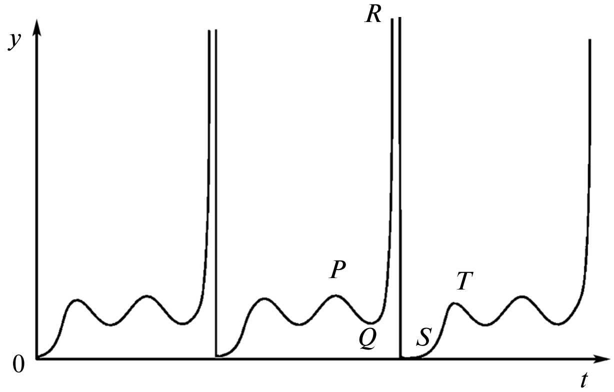

Fig. 10.

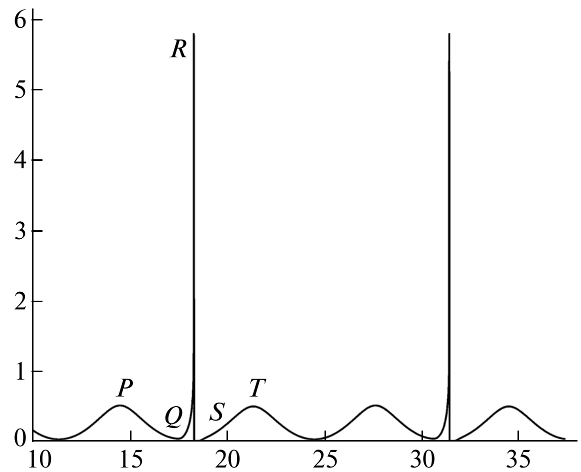

Fig. 11.

A particular example of multidimensional model (1.2) for $x \in \mathbb{R}^3$ is the system in which

$$f_1 = \left((x_1 - b_1)\frac{\rho}{\sqrt{1+\rho^2}} - x_2, x_2\frac{\rho}{\sqrt{1+\rho^2}} + x_1 - b_1, -\chi x_3\right),$$

$$\rho = ((x_2 - b_2)^2 + x_2^2 - b_2^2)^2 + \mu, \tag{4.7}$$

$$f_2 = \alpha(10(x_2 - x_1), 28x_1 - x_2 - 40x_1x_3, -8x_3/3 + 40x_1x_2),$$

$$g(x) = \delta_1 x_1^2 + \delta_2 x_2^2 + \delta_3 x_3^2, \quad h(y) = c_1 y\exp(-y) + c_2(1 - \exp(-y)).$$

A numerical experiment has shown that, at the parameter values

$$b_1 = 0, \quad b_2 = 2, \quad c_1 = 18, \quad c_2 = 1, \quad \alpha = 0.5, \quad \chi = 10, \quad \varepsilon = 0.001, \quad \mu = 0.007,$$
$$\delta_1 = 1, \quad \delta_2 = 0.6, \quad \delta_3 = 1.9, \tag{4.8}$$

system (1.2), (4.7) has a chaotic attractor A with Lyapunov exponents $\lambda_1 \approx 1.1$, $\lambda_2 = 0$, $\lambda_3 \approx -10.02$, and $\lambda_4 \approx -13255$ and Lyapunov dimension $d_L \approx 2.11$ (which is given by the well-known Kaplan–Yorke formula). Figure 10 shows the component y on the attractor A as a function of t plotted on the scale 1 : 10. This scale was chosen for illustrative purposes, since the P, Q, S, and T waves are then better seen. Note that, according to our asymptotic theory, R is asymptotically tall (on the order of $\varepsilon^{-1/2}$).

It should also be noted that, under conditions (4.6), the largest Lyapunov exponent λ_{max} of the chaotic hyperbolic attractor of system (1.2) satisfies the heuristic formula $\lambda_{max} \simeq \ln n_0$. Thus, it seems plausible to hypothesize that, in the case of (4.7) and (4.8), these conditions are satisfied for $n_0 = 3$.

To conclude, we note that all the results obtained remain valid for a system of the form

$$\dot{x} = f_1(x, \mu) + \varepsilon^{p_1} y^{p_2} f_2(x), \quad \varepsilon\dot{y} = g(x) - h(y), \tag{4.9}$$

where the functions f_1, f_2, g, and h are the same as before, while the exponents $p_2 > p_1 > 0$ are arbitrary. Indeed, the transition from (1.2) to (4.9) changes only the equations describing the turning interval. Specifically, after making the substitutions $y = u\varepsilon^{-\beta}$ and $t - t_1(\mu) = \varepsilon^{1-\beta}\theta$ in (1.21), where $\mu = (p_1 + 1)/(p_2 + 1) < 1$, we obtain the Cauchy problem

$$\frac{dx}{d\theta} = u^{p_2} f_2(x), \quad \frac{du}{d\theta} = g(x) - z_{**}, \quad x|_{\theta=0} = x_1(\mu), \quad u|_{\theta=0} = 0. \tag{4.10}$$

Setting $d\tau = u^{p_2}d\theta$ in (4.10), we see that analogues of formulas (1.23) hold true. More precisely, the first of these formulas remains unchanged, while the second becomes

$$u(\tau, \mu) = ((p_2 + 1)a(t, x_0))^{1/(p_2+1)}\Big|_{t=\tau,\, x_0 = x_1(\mu)}.$$

In this paper, the study was based on system (4.9) with $p_1 = 1/2$ and $p_2 = 2$. Recall that, in this case, the oscillations of y are on the order of $\varepsilon^{-1/2}$. However, if necessary, this order can be diminished by choosing suitable p_1 and p_2. For example, if $p_1 = 1/2$ and $p_2 = 5$, then the oscillations of y on a stable cycle of system (4.9), (4.1) with parameter values

$$b_1 = 1, \quad b_2 = 2, \quad c_1 = 28, \quad c_2 = 1, \quad \alpha = 0.5, \quad \delta_1 = \delta_2 = 1, \quad \varepsilon = 0.01, \quad \mu = 0.0015$$

have the form shown in Fig. 11. In this case, the height of R waves is on the order of $\varepsilon^{-1/4}$, which agrees to a higher degree with an actual ECG (see Fig. 7).

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